Perturbation of the Non-Resonance Eigenvalue of a Polyharmonic Matrix Operator

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Abstract
In this paper, we consider a matrix operator
\[ H(l, q)u = (-\Delta)^l u + V(x)u, \]
where \((-\Delta)^l\) is a diagonal \(s \times s\) matrix, whose diagonal elements are the scalar polyharmonic operators, \(V\) is the operator of multiplication by a symmetric \(s \times s\) matrix, \(V(x)\) is periodic with respect to an arbitrary lattice and \(s \geq 2\), \(x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d\), \(d \geq 2\), \(\frac{1}{2} < l < 1\).

We obtain asymptotic formulae of arbitrary order for the non-resonance eigenvalues of this operator.

Keywords: system of polyharmonic operators, periodic, eigenvalue, asymptotic.

1. Introduction
For \(\frac{1}{2} < l < 1\), we consider the operator
\[ H(l, q)u = (-\Delta)^l u + V(x)u \]
in \(L^2(R^d)\), where \((-\Delta)^l\) is a diagonal \(s \times s\) matrix, its diagonal elements being the scalar polyharmonic operators; \(V(x) = (v_{ij}(x))\), \(i, j = 1, 2, ..., s\), is a symmetric \(s \times s\) matrix, \(V = V^T\) and \(s \geq 2\), \(x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d\), \(d \geq 2\).
We suppose that each entry \( v_{ij}(x) \) is a real valued function of \( W^m_2(K) \) and is periodic with respect to the same arbitrary lattice \( \Omega \), \( K = \mathbb{R}^d \backslash \Omega \) is a fundamental domain of \( \Omega \) and \( m > \frac{(4d + 2)(d + 20)}{2} \).

Let \( \Gamma = \{ \gamma \in \mathbb{R}^d : (\gamma, w) \in 2\pi \mathbb{N}, w \in \Omega \} \) be the dual lattice of \( \Omega \) and \( K' = \mathbb{R}^d / \Gamma \) be its fundamental domain. It is well known that the spectral analysis of \( H(l,q) \) can be reduced to studying the operators \( H(l,q) \) defined by the differential expression (1) in \( L^2_z(K) \) and the quasiperiodic condition
\[
u(x + w) = e^{i\nu \cdot t} u(x), \quad w \in \Omega, t \in K',
\]
\[
u(x) = (u_1(x), u_2(x), ..., u_s(x)), \quad x \in K.
\]
Here, \( \cdot \) denotes the innerproduct in \( \mathbb{R}^d \).

The spectrum of the operator \( H(l,q) \) consists of the eigenvalues \( \Lambda_1(t) \leq \Lambda_2(t) \leq ... \) and \( \text{spec}(H(l,q)) = \cup_{n=1}^{m} \{ \Lambda_n(t) : t \in K' \} \). Let \( \Psi_{mj}(x) \) denote the eigenfunction of \( H(l,q) \) corresponding to the eigenvalue \( \Lambda_n(t) \). The eigenvalues of the unperturbed operator \( H(l,0) \) are \( |\gamma + t|^{2d} \) and the corresponding eigenspaces are
\[
E_{\gamma,t} = \text{span}(\Phi_{\gamma,t,1}(x), \Phi_{\gamma,t,2}(x), ..., \Phi_{\gamma,t,m}(x))
\]
\[
\Phi_{\gamma,t,j}(x) = (0, ..., 0, e^{i(\gamma \cdot t + j)x}, 0, ..., 0),
\]
\[
j = 1, 2, ..., s,
\]
for \( \gamma \in \Gamma, t \in K' \). We note that the non-zero component \( e^{i(\gamma \cdot t + j)x} \) of \( \Phi_{\gamma,t,j}(x) \) stands in the \( j \)th component.

It is convenient to define a periodic function \( v_{ij}(x) \) in \( W^m_2(K) \) as a function satisfying the relation
\[
\sum_{\gamma \in \Gamma} |v_{ij}(x)|^2 (1 + |\gamma + t|^{2m}) < \infty,
\]
where
\[
v_{ij}(x) = (v_{ij}(x), e^{i\gamma \cdot x}) = \int_K v_{ij}(x)e^{-i\gamma \cdot x} dx,
\]
(\( \cdot \) is the inner product in \( L_2(K) \)). Moreover, for a big parameter \( \rho \), we can write
\[
v_{ij}(x) = \sum_{\gamma \in \Gamma} v_{ij}(x) e^{i\gamma \cdot x} + O(\rho^{-pa})
\]
and define
\[
M_{ij} = \sum_{\gamma \in \Gamma} |v_{ij}(x)|^2 < \infty,
\]
for all \( i, j = 1, 2, ..., s \), where \( p = m - d \), \( \alpha > 0 \) and \( \Gamma(\rho^{\alpha}) = \{ \gamma \in \Gamma : 0 < |\gamma + t| < \rho^\alpha \} \).

If \( \gamma = 0 \), \( v_{ij0} = \int_K v_{ij}(x) dx \) and \( V_0 = (v_{ij0}) = \int_K V(x)dx \) is a symmetric \( s \times s \) matrix.

The aim of this paper is to obtain the high energy asymptotics of the non-resonance eigenvalues (roughly, the ones far away from the diffraction planes \( \{ x \in \mathbb{R}^d : |x_1^2 + |x + h|^{2d} < \rho \} \) of the operator (1) for arbitrary \( \frac{1}{2} < \rho < 1 \) and arbitrary \( d \geq 2 \), where the potential \( V(x) \) satisfies (3).

Due to its physical importance, the most significant progress has been achieved in the case of the Schrödinger operator, i.e., the case \( l = 1 \) in (1). For the first time asymptotic formulae for the eigenvalues of the periodic (with respect to an arbitrary lattice) Schrödinger operator are obtained in the papers [1-4] by O.A. Veliev. Another proof of asymptotic formulae for quasiperiodic boundary conditions in two and three dimensional cases are obtained in [5, 6, 7, 8]. The asymptotic formulae for the eigenvalues of the Schrödinger operator with periodic boundary conditions are obtained in [9]. When this operator is considered with Dirichlet boundary conditions, the high energy asymptotics of the eigenvalues are obtained in [10]. In papers [11, 12, 13], we obtained the formulae for the eigenvalues of the Schrödinger operator considered with Dirichlet and Neumann boundary conditions on a \( d \)-dimensional parallelepiped, for arbitrary \( d \geq 2 \).

The high energy asymptotics of eigenvalues of \( H(l,q) \) for \( 4l > d + 1 \) \( (d \geq 2) \) are obtained by Yu. Karpeshina in [14] and for arbitrary \( l \geq 1 \) \( (d \geq 2) \) by O.A. Veliev in [15], where he claimed that the assumption \( l \geq 1 \) can be replaced by \( l > n_{m,d} \) for some number \( n_{m,d} < 1 \) that depends on \( m \) (the smoothness of \( q(x) \)) and \( d \) (the dimension) without giving any technical details.

For the matrix case, \( s \geq 1 \), \( d \geq 2 \), \( l \geq 1 \) and \( 4l > d + 1 \), asymptotic formulae for the eigenvalues of the operator (1) are obtained in [16].
In this paper, we obtain the asymptotic formulae of non-resonance eigenvalues of (1) when \( \frac{1}{2} < l < 1 \), \( (\pi m, d) = \frac{1}{2} \), \( s \geq 2 \).

2. Material and Method

We use the same method introduced by O.A. Veliev in his papers [3,4,15] and define the following parameters:

\[
\begin{align*}
\alpha(l) &= \frac{a}{(d+20)3^{r+1}}, \\
\alpha_1(l) &= 3\alpha(l),
\end{align*}
\]

where \( l = \frac{1}{2} + a \), \( 0 < a < \frac{1}{2} \). By these notations (4) becomes

\[
v_{ij}(x) = \sum_{\gamma \in \Gamma(l))} v_{ij}(x) \mu_{\gamma}(x) + O(\rho^{-\alpha(l)}),
\]

where \( \Gamma(l)) = \{ \gamma \in \Gamma, 0 < |\gamma + t| < \rho^{\alpha(l)} \} \), \( p = m - d \) and \( \rho \) is a large parameter.

In the sequel, \( c_1, c_2, c_3, \ldots \) denote the positive constants whose exact values are inessential (they do not depend on \( p \)). Additionally, by \( |a| \sim \rho \), we mean that there exist \( c_1, c_2 \) such that \( c_1 \rho < |a| < c_2 \rho \).

We divide the eigenvalues \( |\gamma + t|^2 \) of the unperturbed operator into two groups. In order to define these groups, we introduce the following sets:

\[
\begin{align*}
V_b^i(\rho^{\alpha(l)}) &= \{ x \in R^d, |x|^2 + |b|^2 < \rho^{\alpha(l)} \}, \\
E_1^i(\rho^{\alpha(l)}, p) &= \bigcup_{b \in \Gamma(p^{\gamma}))} V_b^i(\rho^{\alpha(l)}), \\
U_1^i(\rho^{\alpha(l)}, p) &= R^d \setminus E_1^i(\rho^{\alpha(l)}, p),
\end{align*}
\]

where the intersection \( \bigcap_{i=1}^k E_1^i(\rho^{\alpha(l)}) \) in \( E_1^i \) is taken over \( \gamma_1, \gamma_2, \ldots, \gamma_k \) which are linearly independent vectors and the length of \( \gamma_i \) is not greater than the length of the other vectors in \( \Gamma \cap \gamma_i R \). The set \( U_1^i(\rho^{\alpha(l)}, p) \) is said to be a non-resonance domain and the eigenvalue \( |\gamma + t|^2 \) is called a non-resonance eigenvalue if \( \gamma \in U_1^i(\rho^{\alpha(l)}, p) \). The domains \( V_b^i(\rho^{\alpha(l)}) \), for all \( b \in \Gamma(p^{\gamma})) \), are called resonance domains and the eigenvalue \( |\gamma + t|^2 \) is a resonance eigenvalue if \( \gamma \in V_b^i(\rho^{\alpha(l)}) \).

**Remark**

If \( x \in R^d, |x| \sim \rho \) and \( \gamma_1 \in \Gamma \) then

\[
|x + \gamma_1| \sim \rho \quad \text{and by the Mean Value Theorem}
\]

\[
|x|^2 - |x + \gamma_1|^2 = \xi |x|^2 - |x + \gamma_1|^2 \quad (8)
\]

where \( \xi \sim \rho \). Therefore for \( \frac{1}{2} < l < 1 \), \( U_1^i(\rho^{\alpha(l)}-2i\xi) \) from which we have

\[
\begin{align*}
(n_{i=1}^k V_1^i(\rho^{\alpha(l)})) &= \bigcap_{i=1}^k V_1^i(\rho^{\alpha(l)}) \setminus U_1^i(\rho^{\alpha(l)}-2i\xi), \\
U_1^i(\rho^{\alpha(l)}-2i\xi, p) &= U_1^i(\rho^{\alpha(l)}, p),
\end{align*}
\]

for \( k = 1, 2, \ldots \)

As noted in the Remark 1 of the paper [15], the expression (9) implies that the non-resonance domain \( U_1^i(\rho^{\alpha(l)}, p) \) has asymptotically full measure in \( R^d \) in the sense that

\[
\int_{\mu(U_1^i(\rho^{\alpha(l)}, p) \cap B(0))} e^{\frac{1}{\lambda}} \to 1 \quad \text{as} \quad \rho \to \infty
\]

where \( B(\rho) = \{ x \in R^d, |x| = \rho \} \), if

\[
a_i(l) - 2l + 2d(a(l) - 1 - a(l))
\]

holds. By the definitions (6) of \( a(l) \) and \( a_k(l) \) the condition (10) holds.

From now on, we assume that \( \gamma \in U_1^i(\rho^{\alpha(l)}, p) \) with \( |\gamma + t| \sim \rho \). To prove the asymptotic formulae for eigenvalue \( \Lambda_0(t) \) of the operator \( H(t, q) \), we use the following well-known formula:

\[
(\Lambda_0(t) - |\gamma + t|^2) < \Phi_{\gamma, l, j} >
\]

\[
= < \Phi_{\gamma, l, j}, V(x) \Phi_{\gamma, l, j} >,
\]

(11)

where \( < \cdot, \cdot > \) denotes the inner product in \( L_2^q(K) \). We substitute the decomposition (7) of \( v_{ij}(x) \) into the formula (11) to obtain

\[
(\Lambda_0(t) - |\gamma + t|^2)c(N, i, j)
\]

\[
= \sum_{l=1}^s \sum_{\gamma \in \Gamma(p^{\gamma}))} v_{ij}(c(N, i, j + \gamma)),
\]

where \( c(N, i, j) = < \Phi_{\gamma, l, i}, \Phi_{\gamma, l, j} > \). If we isolate the terms with the coefficient \( c(N, i, j) \); that is, the terms with \( \gamma_1 = 0 \) for each \( i = 1, 2, \ldots, s \), then we get

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\[(A_t - |\gamma + t|^2) c(N, j, y) = \sum_{i=1}^{\infty} v_{ij} c(N, i, y) + \sum_{i=1}^{\infty} \sum_{j, y \in \Gamma, \rho^a(i, j)} c(N, i, y) + O(\rho^{-p a t}) \]

Also, (11) together with (7) imply
\[c(N, j, y) = \frac{\Psi_{N(t)} \Phi_{N(t)}}{\Lambda_N(t) - |\gamma + t|^2} \sum_{i=1}^{\infty} \sum_{j, y \in \Gamma, \rho^a(i, j)} c(N, i, y) + O(\rho^{-p a t}) \]

for every vector \( \gamma \in \frac{\Gamma}{2} \) satisfying the condition
\[|A_t - |\gamma + t|^2| > \frac{1}{2} \rho^a(t) \]

which is called the iterability condition. Note that, if \( \gamma \in U(t(p^a), \rho) \) and

\[|A_t - |\gamma + t|^2| < \frac{1}{2} \rho^a(t), \]

then (14) holds for \( \gamma = y + b \), \( b \in \Gamma(p^a(t)) \). Hence, when \( \gamma \in \Gamma(p^a(t)) \), we may substitute \( \gamma \) for \( \gamma \) in (13) and then the equation (12) becomes

\[(A_t - |\gamma + t|^2) c(N, j, y) = \sum_{i=1}^{\infty} v_{ij} c(N, i, y) + \sum_{i=1}^{\infty} \sum_{j, y \in \Gamma, \rho^a(i, j)} c(N, i, y) + O(\rho^{-p a t}) \]

By isolating the terms with coefficient \( c(N, i, y) \) in the last equation, we obtain

\[(A_t - |\gamma + t|^2) c(N, j, y) = \sum_{i=1}^{\infty} v_{ij} c(N, i, y) + \sum_{i=1}^{\infty} \sum_{j, y \in \Gamma, \rho^a(i, j)} c(N, i, y) + O(\rho^{-p a t}) \]

If we write this equation for \( j = 1, 2, \ldots, s \) and \( i = 1, 2, \ldots, s \), after the first step of the iteration, we obtain the following system:

\[[(A_t - |\gamma + t|^2) I - V_0] A(N, y) = S^t A(N, y) + R^t + O(\rho^{-p a t}) \]

where \( I \) is the \( s \times s \) identity matrix,
\[A(N, y) = (c(N, i, y)),\]
\[S^t = (s^t_j) \text{ is the } s \times s \text{ matrix whose entries are} \]
\[s^t_{ij} = \sum_{l=1}^{s} v_{ij} c(N, i, y + y_i + t^l) \]
\[\text{and } R^t = (r^t_j) \text{ is the vector whose components are} \]

\[r^t_j = \sum_{l=1}^{s} \sum_{i,j} c(N, i, y + y_i + t^l) \sum_{y \in \Gamma} (A(c(N, i, y + y_i + t^l))) - (A(c(N, i, y) + y_i + t^l)) \]

In this way, if we repeat the iteration \( p_1 = |\frac{t^2}{2} \) times and each time we isolate the terms with coefficient \( c(N, i, y) \), we have

\[[(A_t - |\gamma + t|^2) I - V_0] A(N, y) = \sum_{k=1}^{p_1} S^k A(N, y) + R^{p_1} + O(\rho^{-p a t}) \]

where
\[S^k A(N, y) = (S^k_j (A(N, y))), \quad k = 1, \ldots, p_1, \quad j = 1, \ldots, s,\]

\[S^k_j (A(N, y)) = \sum_{i,j}^{s} \sum_{l=1}^{s} v_{ij} c(N, i, y + y_i + t^l) \sum_{y \in \Gamma} (A(c(N, i, y + y_i + t^l))) - (A(c(N, i, y) + y_i + t^l)) \]

and
\[R^{p_1} = (r^{p_1})_j \]

\[r^{p_1} = \sum_{k=1}^{p_1} \sum_{l=1}^{s} \sum_{y} v_{ij} c(N, i, y + y_i + t^l) \sum_{y \in \Gamma} (A(c(N, i, y + y_i + t^l))) - (A(c(N, i, y) + y_i + t^l)) \]
Since the vectors $\gamma_i \in \Gamma(\rho^\alpha(l))$, we have $|b| = |\gamma_1 + \gamma_2 + \ldots + \gamma_1| < \rho_p \rho^\alpha(l)$, for all $i = 1, 2, \ldots, p_\nu$, in (17) and (10). Therefore, (14) together with (5) imply
\[
S^k(A_N(t)) = O(\rho^{-ka_1(t)}), R^k = O(\rho^{-p_1a_1(t)})
\]
for $k = 1, 2, \ldots, p_1$. To obtain (19), we have only used the iterability condition in (14); that is, $\Lambda_N(t) = l$, where $\Lambda_N(t) = |\gamma_1 + \gamma_2 + \ldots + \gamma_1| < \rho_p \rho^\alpha(l)$. Hence, we may conclude that $S^k(a) = O(\rho^{-ka_1(t)})$, $\sum_{s=1}^{p_1} S^s(a) = O(\rho^{-a_1(t)})$, $\forall a \in I$
and
\[
[D(A_N, \gamma) - S(a, p_1)]A_N(t) = O(\rho^{-pa_1(t)}),
\]
where $D(A_N, \gamma) \equiv (A_N(t) - |\gamma + t|^{2l})I - V_0$ and $S(a, p_1) \equiv \sum_{s=1}^{p_1} S^s(a)$. We note that since $V$ is symmetric, $V_0$ and $S(a, p_1)$ are symmetric real valued matrices; hence $D(A_N, \gamma) - S(a, p_1)$ is a symmetric real valued matrix.

We denote the eigenvalues of $V_0$, counted with multiplicity, and the corresponding orthonormal eigenvectors by $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_s$ and $\omega_1, \omega_2, \ldots, \omega_s$, respectively. Thus
\[
V_0 \omega_i = \lambda_i \omega_i, \quad \omega_i \cdot \omega_j = \delta_{ij}.
\]
We let $\beta_i \equiv \beta_i(A_N, \gamma, a)$ denote an eigenvalue of the matrix $D(A_N, \gamma) - S(a, p_1)$ and $f_i \equiv f_i(A_N, \gamma, a)$ its corresponding normalized eigenvector. That is,
\[
[D(A_N, \gamma) - S(a, p_1)]f_i = \beta_i f_i,
\]
where $f_i \cdot f_j = \delta_{ij}, i, j = 1, 2, \ldots, s$.

3. Results

**Lemma 1** Suppose $\frac{1}{2} < l < 1$, $\gamma \in U(\rho^\alpha(l), p)$ and $|\gamma + t| \sim \rho$.

(a) Let $\beta_i$ be an eigenvalue of the matrix $D(A_N, \gamma) - S(a, p_1)$ and $f_i = (f_{i_1}, f_{i_2}, \ldots, f_{i_s})$ its corresponding normalized eigenvector. Then there exists an integer $N \equiv N_1$ such that $\Lambda_N(t)$ satisfies (15) and
\[
|A(N, \gamma) \cdot f_i| > c_4 \rho^{-\frac{(d-1)}{2}}.
\]
(b) Let $\Lambda_N(t)$ be an eigenvalue of the operator $H_c(t, \gamma)$ satisfying the inequality (15). Then there exists an eigenfunction $\Psi_{\gamma, t}(x)$ of the operator $H_c(t, 0)$ such that
\[
|c(N, t, \gamma)| > c_4 \rho^{-\frac{(d-1)}{2}}.
\]

**Proof. (a)** By a well-known result from perturbation theory, the $N$th eigenvalue of the operator $H_c(t, \gamma)$ lies in $M$-neighborhood of the $N$th eigenvalue of the operator $H_c(t, 0)$; that is, there is an integer $N$ such that
\[
|\Lambda_N(t) - |\gamma + t|^{2l}| < \rho^{\alpha_1(2l)}.
\]
On the other hand, since $H_c(t, \gamma)$ is a self-adjoint operator, the eigenfunctions $(\Psi_{\gamma, t}(x))_{N=1}^\infty$ of $H_c(t, \gamma)$ form an orthonormal basis for $L_2^2(\mathbb{R})$. By using Parseval’s relation, we have
\[
\begin{align*}
\sum_{N=1}^\infty |\Lambda_N(t) - |\gamma + t|^{2l}|^2 |c_2^p a_1(t)|^2 &< \sum_{N=1}^\infty \sum_{j=1}^s |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2 \\
&< \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2 \leq \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2 \\
&= \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2 \leq \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2
\end{align*}
\]
Now, we estimate the last expression in (25). By using the Cauchy-Schwartz inequality and (11), we get
\[
\begin{align*}
\sum_{N=1}^\infty |\Lambda_N(t) - |\gamma + t|^{2l}|^2 |c_2^p a_1(t)|^2 &< \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2 \\
&< \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2 \\
&< \sum_{j=1}^s \sum_{N=1}^\infty |f_i \Phi_{\gamma, t, j} \Psi_{N, t} > |^2
\end{align*}
\]
By Parseval’s relation, we have

$$
H(t)
$$

perturbation theory, the number of eigenvalues (5) we get

$$
\text{for a fixed } \gamma \text{ satisfying } (15) \text{; that is, }
$$

$$
A(N, \gamma) \cdot f_i^2 = 1 - O(\rho^{-2\alpha(i)}). \quad \text{(26)}
$$

On the other hand, if \( \alpha \sim \rho \), then the number of

$$
y \in \mathbb{R}_+ \text{ satisfying } |y - a^2| < 1 \text{ is less than }
$$

$$
c_\rho \rho^{d-1}. \quad \text{Therefore, the number of eigenvalues of }
$$

$$
H_i(t, 0) \text{ lying in } (a^2 - 1, a^2 + 1) \text{ is less than } c_\rho \rho^{d-1}. \text{ By this result and the result of }
$$

$$
\text{perturbation theory, the number of eigenvalues } 
$$

$$
A_N(t) \text{ of } H_i(t, V) \text{ in the interval } [|y + t|^{2l} - 
$$

$$\frac{1}{2} \rho^{a(i)}, |y + t|^{2l} + \frac{1}{2} \rho^{a(i)}] \text{ is less than } c_\rho \rho^{d-1}. \text{ Thus }
$$

$$
1 - O(\rho^{-2\alpha(i)}) = 
$$

$$
\sum_{N: |A_N(t) - \gamma + t|^{2l} < \frac{1}{2} \rho^{a(i)}} |A(N, \gamma) \cdot f_i^2| < c_\rho \rho^{d-1} |A(N, \gamma) \cdot f_i^2| \quad \text{(27)}
$$

from which we get (23).

(b) Since \( H_i(t, 0) \) is a self-adjoint operator the set of
eigenfunctions \( \{ \Phi_{\gamma, i}(x) \}_{\gamma \in \mathbb{R}_+} \) of \( 
H_i(t, 0) \) forms an orthonormal basis for \( L^2(K) \).

By Parseval’s relation, we have

$$
\sum_{\gamma: |A_{\gamma}(t) - |\gamma + t|^{2l}| |c_{\rho^{a(i)}} \sum_{i=1}^z |c(N, i, \gamma)|^2
$$

$$
= \sum_{\gamma: |A_{\gamma}(t) - |\gamma + t|^{2l}| |c_{\rho^{a(i)}} \sum_{i=1}^z |c(N, i, \gamma)|^2
$$

We estimate the last expression in (28). Hence for a fixed \( i = 1, 2, ..., s \), using (11) together with (5) we get

$$
\sum_{\gamma: |A_{\gamma}(t) - |\gamma + t|^{2l}| |c_{\rho^{a(i)}} \sum_{i=1}^z |c(N, i, \gamma)|^2
$$

From the last equality and (28), we obtain

$$
\sum_{\gamma: |A_{\gamma}(t) - |\gamma + t|^{2l}| |c_{\rho^{a(i)}} \sum_{i=1}^z |c(N, i, \gamma)|^2
$$

Arguing as in the proof of (a), we get

$$
1 - O(\rho^{-2\alpha(i)}) = \sum_{\gamma: |A_{\gamma}(t) - |\gamma + t|^{2l}| |c_{\rho^{a(i)}} \sum_{i=1}^z |c(N, i, \gamma)|^2
$$

from which (24) follows.

**Theorem 2** Suppose \( \frac{1}{2} < l < 1, y \in U(\rho^{a(i)}, p) \) and \( |y + t| \sim \rho \).

(a) For each eigenvalue \( \lambda_i \) of the matrix \( V_0 \), there exists an eigenvalue \( \Lambda_N(t) \) of the operator \( H_i(t, V) \) satisfying

$$
\Lambda_N(t) = |y + t|^{2l} + \lambda_i + O(\rho^{-\alpha(i)}).
$$

(b) For each eigenvalue \( \Lambda_N(t) \) of the operator \( H_i(t, V) \) satisfying (15), there exists an eigenvalue \( \lambda_i \) of the matrix \( V_0 \) satisfying (30).
Multiplying both sides of the above equation by \( f_i \) gives
\[
\beta_i [A(N, \gamma) \cdot f_i] = O(\rho^{-\frac{d-1}{2a(i)}}).
\]
By using the inequality (23) in the above equation, we get
\[
\beta_i = O(\rho^{-\frac{d-1}{2a(i)}}).
\]
(31)

Since \( D(A_N, \gamma) \) and \( S(A_N, p_1) \) are symmetric real valued matrices, by a well known result in matrix theory (see [13]),
\[
|\beta_i - (A_N(t) - |\gamma + t|^{2t} - \lambda_i)| \leq \| S(A_N, p_1) \| 
\]
which together with (18) imply that
\[
\beta_i = A_N(t) - |\gamma + t|^{2t} - \lambda_i + O(\rho^{-\frac{d-1}{2a(i)}}).
\]
(32)

Hence, by choosing \( p > \frac{d-1}{2a(i)} + 1 \) and using (32) and (31), we get the result.

(b) By Lemma (1h), there exists \( \varPhi_{T,1}(x) \) satisfying (24) from which we have
\[
\| A(N, \gamma) \| > \epsilon_0 \rho^{-\frac{d-1}{2a(i)}}.
\]
(33)

Now, we consider the equation (16) for these \((N, \gamma)\) pairs
\[
|\| A(N(t) - |\gamma + t|^{2t})| - V_0 \| |A(N, \gamma) = S(A_N, p_1)A(N, \gamma) + O(\rho^{-\frac{d-1}{2a(i)}}).
\]

First, we apply
\[
\frac{1}{4\| A(N, \gamma) \|} \| (A_N(t) - |\gamma + t|^{2t})| - V_0 \|^{-1}
\]
to both sides of the above equation. Next, we take the norm of both sides and use (33) to obtain the following inequality
\[
1 \leq \| (A_N(t) - |\gamma + t|^{2t})| - V_0 \|^{-1} \| \sum_{k=1}^{p_1} S_k \| \| (A_N(t) - |\gamma + t|^{2t})| - V_0 \|^{-1} \| O(\rho^{-\frac{d-1}{2a(i)}}).
\]

By estimation (20) and choosing \( p > \frac{d-1}{2a(i)} + 1 \), we get
\[
1 \leq \max_{i=1, \ldots, 3} \| A_N(t) - |\gamma + t|^{2t} - \lambda_i \| \leq 1 \| O(\rho^{-\frac{d-1}{2a(i)}})
\]

where minimum (maximum) is taken over all eigenvalues of the matrix \( V_0 \) from which we obtain the result.

In the interest of saving space, we use the notation
\[
a_{\gamma,k} = |\gamma + t|^{2t} + \lambda_k + \| F_{j-1} \|
\]
where
\[
F_0 = 0, \quad F_j = S\{((\gamma + t)|^{2t} + \lambda_k)\}, \quad F_j = S(a_{\gamma,k,j}), \quad j \geq 2.
\]
(34)

Then, we have
\[
\| F_j \| = O(\rho^{-\frac{d-1}{2a(i)}})
\]
(35)

for all \( j = 1, 2, \ldots, p - c, \quad c = \frac{d-1}{2a(i)} + 1 \). Indeed, since \( F_0 = 0, \quad F_j \| \| = 0 \) and if we assume that
\[
\| F_{j-1} \| = O(\rho^{-\frac{d-1}{2a(i)}}), \quad \text{then since } a_{\gamma,k} \in I, \quad \text{by (20), we have } \| F_j \| = O(\rho^{-\frac{d-1}{2a(i)}}).
\]

By (35), we have \( a_{\gamma,k} + O(\rho^{-\frac{d-1}{2a(i)}}) \in I \). Thus, we let \( a \equiv a_{\gamma,k} + O(\rho^{-\frac{d-1}{2a(i)}}) \) in (20), to get
\[
[D(A_N, \gamma) - S(a_{\gamma,k} + O(\rho^{-\frac{d-1}{2a(i)}}), p_1)]A(N, \gamma) = O(\rho^{-\frac{d-1}{2a(i)}}).
\]
(36)

We add and subtract the term \( F_j A(N, \gamma) = S(a_{\gamma,k,j})A(N, \gamma) \) into the left hand side of the equation (36) to obtain
\[
[D(A_N, \gamma) - F_j A(N, \gamma) - E_j A(N, \gamma) = O(\rho^{-\frac{d-1}{2a(i)}}).
\]
(37)

where
\[
E_j = [S\{a_{\gamma,k} + O(\rho^{-\frac{d-1}{2a(i)}}), f\} - S(a_{\gamma,k,j})] + \sum_{l=1}^{p_1} S_k(a_{\gamma,k} + 1 \| F_{j-1} \| O(\rho^{-\frac{d-1}{2a(i)}})).
\]

By (20), we have
\[
\sum_{l=1}^{p_1} S_k(a_{\gamma,k} + O(\rho^{-\frac{d-1}{2a(i)}}), f) - S(a_{\gamma,k,j}) \| = O(\rho^{-\frac{d-1}{2a(i)}}).
\]
(38)

If we prove that
\[
\| S(a_{\gamma,k} + O(\rho^{-\frac{d-1}{2a(i)}}), f) - S(a_{\gamma,k,j}) \| = O(\rho^{-\frac{d-1}{2a(i)}}),
\]
(39)
then it follows from (38) and (39) that

\[ \| E_j \| = O(\rho^{-(j+1)\alpha_1(0)}). \]  

(40)

Suppose \( a_{r,k} \in I \), we have

\[ |a_{r,k} + O(\rho^{-ja_1(0)}) - |y + y_1 + \ldots + y_t + t|^2| > \frac{1}{2} \rho^{a_1(t)}, \]

\[ |a_{r,k} - |y + y_1 + \ldots + y_t + t|^2| > \frac{1}{2} \rho^{a_1(t)}, \]

(41)

for all \( y \in \Gamma(a_1(0)) \) and \( t = 1,2,\ldots,p_k \). We first calculate the order of the first term of the summation in (39). To do this, we consider each entry of this term, and use (41) and (5):

\[ |s^2_n(a_{r,k} + O(\rho^{-ja_1(0)}) - s^2_n(a_{r,k})| \]

\[ \leq \sum_{n=1}^{\infty} \sum_{n_1,n_2 \in \Gamma(a_0(0))} |v_{n,n_1}| |v_{n,n_2}| |O(\rho^{-ja_1(0)})|

\[ (a_{r,k} + O(\rho^{-ja_1(0)}) - |y + y_1 + t|^2)(a_{r,k} - |y + y_1 + t|^2)] \]

\[ \leq c_{11} \rho^{-(j+2)a_1(0)}, \]

for each \( n, i = 1,2,\ldots,s \), which implies

\[ \| S^i(a_{r,k} + O(\rho^{-ja_1(0)})) - S^i(a_{r,k}) = O(\rho^{-ja_1(0)}) \]

Therefore, by direct calculations, it can be easily seen that

\[ \| S^i(a_{r,k} + O(\rho^{-ja_1(0)})) - S^i(a_{r,k}) = O(\rho^{-ja_1(0)}) \]

from which we obtain (39).

**Theorem 3** Suppose \( \frac{1}{2} < l < 1 \), \( y \in \Gamma(a(0),p) \) and \( |y + t| \sim \rho \).

(a) For any eigenvalue \( \lambda_i \), \( i = 1,2,\ldots,s \) of the matrix \( V_0 \), there exists an eigenvalue \( \Lambda_N(t) \) of the operator \( H_t(L, V) \) satisfying the following formula:

\[ \Lambda_N(t) = |y + t|^2 + \lambda_i + \| F_{k-1} \| + O(\rho^{-ka_1(t)}), \]

(42)

where \( F_{k-1} \) is given by (34), \( k = 1,2,\ldots,p - c \).

(b) For any eigenvalue \( \Lambda_N(t) \) of the operator \( H_t(L, V) \) satisfying (15), there is an eigenvalue \( \lambda_i \) of the matrix \( V_0 \) satisfying (42).

**Proof (a)** By Lemma(1a), there exist \( \Lambda_N(t) \) and \( \Psi_{k,t}(x) \) satisfying (15) and (23), respectively. We prove the theorem by induction. For \( k = 1 \), we obtain the result by Theorem(2a).

Now, assume that for \( k = j - 1 \) the formula (42) is true; that is,

\[ \Lambda_N(t) = |y + t|^2 + \lambda_i + \| F_{j-1} \| + O(\rho^{-ja_1(0)}). \]

(43)

Let \( \beta_i \) be an eigenvalue of the matrix \( D(\Lambda_N(t)) = S((a_{r,k} + O(\rho^{-ja_1(0)}),p_k) \). If we multiply both sides of the equation (36) by its corresponding normalized eigenvector \( f_i \) and use (23), then we obtain

\[ \beta_i = O(\rho^{-ja_1(0)}). \]

(44)

On the other hand, the matrix

\[ D(\Lambda_N(t)) = S((a_{r,k} + O(\rho^{-ja_1(0)}),p_k) \]

in (36) is decomposed as follows

\[ D(\Lambda_N(t)) = S((a_{r,k} + O(\rho^{-ja_1(0)}),p_k) \]

\[ = D(\Lambda_N(t) - F_j = E_j. \]

Thus, by (40), (44) and a well known result in matrix theory,

\[ \beta_i - \Lambda_N(t) - (\gamma + t|^2 + \lambda_i) \]

\[ \leq \| F_j \| + O(\rho^{-ja_1(0)}), \]

where \( 1 \leq j + 1 \leq p - c \), we get the proof of (42).

(b) Again, we prove this part of the theorem by induction. For \( j = 1 \), we obtain the result by Theorem(2b).

Now, assume that for \( k = j - 1 \) the formula (42) is true. To prove (42) for \( k = j \), we use the equation (37) and the definition of the matrix \( D(\Lambda_N(t)) \) and get

\[ [(\Lambda_N(t) - |y + t|^2)I - D_j]A(N,y) \]

\[ = E_j A(N,y) + O(\rho^{-ja_1(t)}), \]

where \( D_j = V_0 + F_j \).

First, we apply \( \frac{1}{E_j}[(\Lambda_N(t) - |y + t|^2)I - D_j]^{-1} \) to both sides of the above equation and then, take the norm of both sides and use the estimations (33) and (40) to obtain

\[ 1 \leq \]

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\[ \| (\Lambda_N(t) - |\gamma + t|^2I - D_i)^{-1} \| \| O(\rho^{-i(j+1)}|\alpha(i)|) \]

\[ \| (\Lambda_N(t) - |\gamma + t|^2I - D_i)^{-1} \| \| O(\rho^{-(p-c)|\alpha(i)|}) \]

\[ \leq \max_{i=1,2,\ldots,s} \frac{1}{|\Lambda_N(t) - |\gamma + t|^2 - \lambda_i(j)|} \| O(\rho^{-i(j+1)}|\alpha(i)|) \],

or

\[ \min_{i=1,2,\ldots,s} |\Lambda_N(t) - |\gamma + t|^2 - \lambda_i(j)| \]

\[ \leq c_1 \rho^{-i(j+1)}|\alpha(i)| \]

where minimum is taken over all eigenvalues \( \lambda_i(j) \) of the matrix \( D_j \), \( 1 \leq j + 1 \leq p - c \). By the last inequality and the well known result in matrix theory, \( |\lambda_i(j) - \lambda_j| \leq \| F_j \| \) and the result follows.

References


