

You will not get any points if your answer is wrong, that is no points to your explanations if your answer is wrong. And of course no points to a correct answer if your explanation or proof is not correct or clear. YOU must write GOOD Mathematics and GOOD luck.

1. Let $A = \{1, 2, 4, 5, 7, 11, 13\}$. Define a relation \mathcal{R} on A by writing $(x, y) \in \mathcal{R}$ if and only if $x - y$ is a multiple of 3.

(a) Show that \mathcal{R} is an equivalence relation on A .

(b) List the equivalence classes of \mathcal{R} .

SOLUTION:

a) \mathcal{R} is reflexive since $a - a = 0$ is a multiple of 3 (3×0)
 \mathcal{R} is symmetric since if $a - b = 3k$ then $b - a = 3(-k)$.
 \mathcal{R} is transitive since if $a - b = 3k_1$ and $b - c = 3k_2$ then $a - c = 3(k_1 + k_2)$, a multiple of 3.

b) $[1] = \{1, 4, 7, 13\}$
 $[2] = \{2, 5, 11\}$

2. Give as good a big- O estimate as possible for each of these functions

(a) $(n^2 + 1)(n + 1)$

(b) $(n \log n + n^2)(n^3 + 1)$

(c) $(n! + 2^n)(n^3 + 2 \log n)$

SOLUTION:

a) $(n^2 + 1)(n + 1) = n^3 + n^2 + n + 1 \leq 4n^3$ for $n > 1$ thus it is $O(n^3)$

b) $(n \log n + n^2)(n^3 + 1) = n^4 \log n + n \log n + n^5 + n^2 \leq 4n^5$ since $\log n < n$, Thus it is $O(n^5)$

c) $n^3 n! + 2n! \log n + n^3 2^n + 2^{n+1} \log n \leq 5n^3 n!$
 since $n^3 > \log n$ and $n! > 2^n$ for $n \geq 4$. So $O(n^3 n!)$

3. Let \mathcal{R} be the relation represented by the matrix

$$M_{\mathcal{R}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix representing a) \mathcal{R}^{-1} b) $\overline{\mathcal{R}}$ c) \mathcal{R}^2 .

SOLUTION:

$$\text{a) } M_{\mathcal{R}^{-1}} = M_{\mathcal{R}}^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{b) } M_{\overline{\mathcal{R}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{\mathcal{R}^2} = M_{\mathcal{R}} \odot M_{\mathcal{R}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

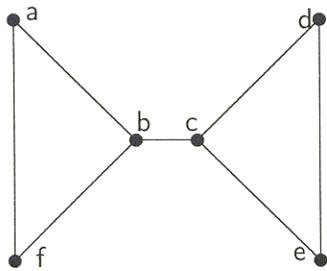
4. Let F be the Boolean function $F(x, y, z) = x + yz$.

- Use a table to express the values of F
- Find the sum-of-product-expansion of F

x	y	z	$x + yz$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

$$\text{b) } x + yz = xy\bar{z} + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz.$$

5. Let G be the following graph



- (a) Find the adjacency matrix
 (b) Does G have an Eulerian circuit and an Eulerian path? Why?

SOLUTION:

$$\begin{array}{c}
 \begin{array}{cccccc}
 & a & b & c & d & e & f \\
 a & 0 & 1 & 0 & 0 & 0 & 1 \\
 b & 1 & 0 & 1 & 0 & 0 & 1 \\
 c & 0 & 1 & 0 & 1 & 1 & 0 \\
 d & 0 & 0 & 1 & 0 & 1 & 0 \\
 e & 0 & 0 & 1 & 1 & 0 & 0 \\
 f & 1 & 1 & 0 & 0 & 0 & 0
 \end{array}
 \end{array}$$

b) No Eulerian circuit
 since $\deg(b) = 3$ (or $\deg(c) = 3$)
 is odd.

Eulerian path exists;

$$c \rightarrow d \rightarrow e \rightarrow c \rightarrow b \rightarrow a \rightarrow f \rightarrow b$$

start either from b or c end
 with the other.

6. (a) Find $f(n)$ when $n = 3^k$, where f satisfies the recurrence relation $f(n) = 2f(n/3) + 4$ with $f(1) = 1$.

- (b) Prove your formula for $f(n)$ by induction on k .

SOLUTION:

$$\begin{aligned}
 f(3^k) &= 2f(3^{k-1}) + 4 \\
 &= 2(2f(3^{k-2}) + 4) + 4 = 2^2 f(3^{k-2}) + 2 \cdot 4 + 4 \\
 &= 2^3 f(3^{k-3}) + 2^2 \cdot 4 + 2^1 \cdot 4 + 4
 \end{aligned}$$

and hence

$$\begin{aligned}
 f(3^k) &= 2^k f(3^{k-k}) + 2^{k-1} \cdot 4 + \dots + 2^1 \cdot 4 + 4 \\
 &= 2^k + 4(1 + 2 + \dots + 2^{k-1}) = 2^k + 4 \cdot (2^k - 1) \\
 &= 5 \cdot 2^k - 4 //
 \end{aligned}$$

b) $f(3) = 6 = 2f(1) + 4$

$$\begin{aligned}
 f(3^{k+1}) &= 2f(3^k) + 4 = 2 \cdot (5 \cdot 2^k - 4) + 4 \\
 &= 5 \cdot 2^{k+1} - 8 + 4 = 5 \cdot 2^{k+1} - 4 //
 \end{aligned}$$

7. (Bonus problem, you are not obliged to solve it !)

Words of length n , using only 3 letters a, b, c are to be transmitted over a communication channel subject to the condition that no word in which two a 's appear consecutively is to be transmitted. Determine the number of words allowed by the communication channel.

Let w_n denote the number of allowed words of length n . We have $w_0 = 1$ (empty word) and

$$w_1 = 3.$$

words starting with a , i.e. a $\underbrace{\hspace{2cm}}$: $2w_{n-2}$ # words
 \downarrow \nearrow
 $b \text{ or } c$ w_{n-2}

" " b $\underbrace{\hspace{2cm}}$: w_{n-1} # words.

c $\underbrace{\hspace{2cm}}$: w_{n-1} # words.

Thus $w_n = 2w_{n-1} + 2w_{n-2}$, $n \geq 2$.

The characteristic equation is

$$x^2 - 2x - 2 = 0 \quad \text{and the roots are}$$

$$r_{1,2} = 1 \pm \sqrt{3}$$

Thus the general solution is

$$w_n = c_1 (1 + \sqrt{3})^n + c_2 (1 - \sqrt{3})^n$$

Putting $n=0$, $c_1 + c_2 = 1$
 $n=1$ $c_1(1 + \sqrt{3}) + c_2(1 - \sqrt{3}) = 3$

Hence $c_1 = \frac{2 + \sqrt{3}}{2\sqrt{3}}$, $c_2 = \frac{-2 + \sqrt{3}}{2\sqrt{3}}$ and therefore

$$w_n = \frac{2 + \sqrt{3}}{2\sqrt{3}} (1 + \sqrt{3})^n + \frac{-2 + \sqrt{3}}{2\sqrt{3}} (1 - \sqrt{3})^n, \quad n \geq 0.$$