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q-Bernstein polynomials and Bézier curves

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Abstract

We define q-Bernstein polynomials, which generalize the classical Bernstein polynomials, and show that the difference of two consecutive q-Bernstein polynomials of a function f can be expressed in terms of second order divided differences of f. It is also shown that the approximation to a convex function by its q-Bernstein polynomials is one sided.

A parametric curve is represented using a generalized Bernstein basis and the concept of total positivity is applied to investigate the shape properties of the curve. We study the nature of degree elevation and degree reduction for this basis and show that degree elevation is variation diminishing, as for the classical Bernstein basis.

Key Words: Generalized Bernstein polynomial; shape preserving; total positivity; degree elevation

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1. Introduction

When representing a parametric curve or surface it is important which basis is used if we wish to preserve the shape of the curve or surface. For these reasons the Bernstein-Bézier curve and surface representation play a significant role in CAGD. See, for example, [5, 11]. In this paper we generalize some of the very well known Bézier curve techniques by using a generalization of the Bernstein basis, called the q-Bernstein basis. The Bézier curve is retrieved when we set the parameter q to the value 1. This paper is organized as follows. First we define a one-parameter family of generalized Bernstein polynomials (called q-Bernstein polynomials) from which we recover the classical Bernstein polynomials when

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we set $q = 1$. We prove that the approximation to a convex function by its q-Bernstein polynomials is one sided. Then we show that the difference of two consecutive q-Bernstein polynomials has a representation involving second order divided differences. We describe some of the shape preserving properties which the generalized Bernstein polynomials share with their classical counterparts. The connection between the power basis, the Bernstein basis and the q-Bernstein basis is revealed by deriving their transformation matrices. We then construct parametric curves using the q-Bernstein basis and discuss shape properties using the concept of total positivity. Finally, we present a degree elevation algorithm for q-Bernstein parametric curves and show that this process is variation diminishing, as in the classical case.

The q-Bernstein polynomials were defined as follows by the second author [15]:

$$\mathcal{B}_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \tag{1.1}$$

where an empty product denotes 1, the parameter q is a positive real number and $f_r = f([r]/[n])$. Here $[r]$ denotes a q–integer, defined by

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$, which is also called a Gaussian polynomial, is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n] \cdot [n-1] \cdots [n-r+1]}{[r] \cdot [r-1] \cdots [1]}$$

for $n \geqslant r \geqslant 1$, and has the value 1 when $r = 0$ and the value zero otherwise. Note that this reduces to the usual binomial coefficient when we set $q = 1$. It satisfies the recurrence relations

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix} \tag{1.2}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}, \tag{1.3}$$

and it can easily be verified by induction on n that

$$(1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} x^r. \tag{1.4}$$

The q-binomial coefficient can be interpreted combinatorially as the generating function for counting restricted partitions. We may write

$$\begin{bmatrix} n \\ r \end{bmatrix} = \sum_{i=0}^{r(n-r)} p(n-r, r, i) q^i,$$

where $p(n-r, r, i)$ is the number of partitions of i with at most r parts each not exceeding $n-r$. It is also related (see [1]) to the problem of counting the number of subspaces over a finite field. We note that $\mathcal{B}_n$, defined by (1.1), is a monotone linear operator for any $0 < q \leqslant 1$ and $\mathcal{B}_n$ reproduces linear functions, that is

$$\mathcal{B}_n(ax+b;x) = ax+b,\ a, b \in \mathbb{R}.$$

It also satisfies the end point interpolation conditions $\mathcal{B}_n(f;0) = f(0)$ and $\mathcal{B}_n(f;1) = f(1)$. It is shown in [14] that (1.1) may be evaluated by the following de Casteljau type algorithm:

Given: $f_0^{[0]}, f_1^{[0]}, \ldots, f_n^{[0]}$
for $m = 1$ **to** n **do**
  for $r = 0$ **to** $n - m$ **do**
    $f_r^{[m]} := (q^r - q^{m-1}x)f_r^{[m-1]} + xf_{r+1}^{[m-1]}$
  return
return

Note that with $q = 1$ we recover the original de Casteljau algorithm. With $q = 1$ and any point $\mathsf{b}_i \in \mathbb{R}^2, \mathbb{R}^3$, this algorithm has a nice geometric interpretation which is called subdivision, see [5].

The effect of introducing the parameter q into the de Casteljau algorithm can be seen in Figure 1 in which surfaces are made by revolving the curves about an appropriate vertical axis.

Figure 1: Seven control points and change of q, $q = 0.8$, $q = 1$.

The generalized Bernstein polynomial $\mathcal{B}_n(f;x)$ defined by (1.1) shares the well–known shape–preserving properties of the classical Bernstein polynomial. Thus when the function f is convex then (see [12]) $\mathcal{B}_{n-1}(f;x) \geqslant \mathcal{B}_n(f;x)$ for $n \geqslant 2$ and any $0 < q \leqslant 1$. In addition, it behaves in a very nice way when we vary the parameter q : it is proved in [10] that $\mathcal{B}_n^r(f;x) \leqslant \mathcal{B}_n^q(f;x)$ for any $0 < q \leqslant r \leqslant 1$. As a consequence of this we can show that the approximation to a convex function by its q-Bernstein polynomial is one sided.

Theorem 1.1 *If f is a convex function on $[0,1]$ then $\mathcal{B}_n(f;x) \geqslant f(x)$ for $0 < q \leqslant 1$.*

Proof Let $l(x)=ax+b$ be any line. Also let l be tangent at an arbitrary point $t\in[0,1]$ so that $l(t)=f(t)$ and $f-l\geqslant 0$. Using $\mathcal{B}_n(ax+b;x)=ax+b$ and the fact that $\mathcal{B}_n$ is a monotone linear operator, we see that $\mathcal{B}_n(f-l)=\mathcal{B}_nf-l\geqslant 0$. Thus, $\mathcal{B}_n(f;t)\geqslant l(t)=f(t)$ at any tangent point t. By continuity we deduce that $\mathcal{B}_nf\geqslant f$. ■

Theorem 1.2 *For $n=2,3,\ldots$ we have*

$$\mathcal{B}_{n-1}(f;x)-\mathcal{B}_n(f;x)=\frac{x(1-x)}{[n-1][n]}\sum_{r=0}^{n-2}q^{n+r-1}\begin{bmatrix}n-2\\ r\end{bmatrix}$$

$$f\left[\frac{[r]}{[n-1]},\frac{[r+1]}{[n]},\frac{[r+1]}{[n-1]}\right]x^r\prod_{s=1}^{n-r-2}(1-q^sx). \tag{1.5}$$

Proof It is shown in [12] that the difference of consecutive q-Bernstein polynomials can be written as

$$\mathcal{B}_{n-1}(f;x)-\mathcal{B}_n(f;x)=\sum_{r=1}^{n-1}\begin{bmatrix}n\\ r\end{bmatrix}x^r a_r\prod_{s=0}^{n-r-1}(1-q^sx), \tag{1.6}$$

where

$$a_r=\frac{[n-r]}{[n]}f\left(\frac{[r]}{[n-1]}\right)+q^{n-r}\frac{[r]}{[n]}f\left(\frac{[r-1]}{[n-1]}\right)-f\left(\frac{[r]}{[n]}\right). \tag{1.7}$$

Let us evaluate the divided difference of f at the points $\frac{[r-1]}{[n-1]}$, $\frac{[r]}{[n]}$ and $\frac{[r]}{[n-1]}$. Using the symmetric form for the divided differences we obtain

$$\begin{aligned} f\left[\frac{[r-1]}{[n-1]},\frac{[r]}{[n]},\frac{[r]}{[n-1]}\right] &= \\ \frac{[n-1]^2[n]}{q^{2r-2}\,[n-r]}f\left(\frac{[r-1]}{[n-1]}\right) &- \frac{[n-1]^2[n]^2}{q^{n+r-2}\,[r][n-r]}f\left(\frac{[r]}{[n]}\right) \\ &+ \frac{[n-1]^2[n]}{q^{n+r-2}\,[r]}f\left(\frac{[r]}{[n-1]}\right) \end{aligned} \tag{1.8}$$

From (1.8) and (1.7) we see that

$$\begin{bmatrix}n\\ r\end{bmatrix}a_r=\frac{q^{n+r-2}}{[n-1][n]}\begin{bmatrix}n-2\\ r-1\end{bmatrix}f\left[\frac{[r-1]}{[n-1]},\frac{[r]}{[n]},\frac{[r]}{[n-1]}\right]$$

and also it follows from (1.6) that

$$\mathcal{B}_{n-1}(f;x)-\mathcal{B}_n(f;x)=\frac{x(1-x)}{[n-1][n]}\sum_{r=1}^{n-1}q^{n+r-2}\begin{bmatrix}n-2\\ r-1\end{bmatrix}$$

$$f\left[\frac{[r-1]}{[n-1]},\frac{[r]}{[n]},\frac{[r]}{[n-1]}\right]x^{r-1}\prod_{s=1}^{n-r-1}(1-q^sx).$$

Shifting the limits of the latter equation completes the proof. ■

The recent study [13] investigates convergence properties of (1.1) as well convergence of its iterates and its Boolean sums.

2. Totally positive bases and the shape of curves

The basis functions which appear in (1.1),

$$B_i^n(x) = \begin{bmatrix} n \\ i \end{bmatrix} x^i \prod_{j=0}^{n-i-1} (1 - q^j x), \quad i = 0, 1, \ldots, n, \tag{2.1}$$

satisfy the following recurrence relations which can be deduced using (1.2) and (1.3) respectively,

$$B_i^n(x) = q^{n-i} x B_{i-1}^{n-1}(x) + (1 - q^{n-i-1}x) B_i^{n-1}(x),$$

and

$$B_i^n(x) = x B_{i-1}^{n-1}(x) + (q^i - q^{n-1}x) B_i^{n-1}(x).$$

It can also be easily verified that

$$B_i^m(x) B_j^n(q^{m-i}x) = \frac{\left[{m \atop i} \right] \left[{n \atop j} \right] q^{(m-i)j}}{\left[{m+n \atop i+j} \right]} B_{i+j}^{m+n}(x).$$

It is shown in [10] that the basis (2.1) provides a normalized totally positive basis (NTP) for $0 < q \leqslant 1$ on the interval $[0, 1]$ for P_n, the space of polynomials of degree not exceeding n. When $q = 1$, (2.1) is simply the classical Bernstein basis. NTP bases such as the generalized Ball basis, the Bernstein basis, and the B-spline basis, have an important role in geometric design which we will mention below.

Let us recall that a matrix is said to be totally positive (TP) if all its minors are non-negative. It is proved in [2] that a finite matrix is TP if and only if it is a product of 1-banded matrices with non-negative elements. We say that a sequence $\Phi = (\phi_0, \ldots, \phi_n)$ of real-valued functions on an interval I is TP if, for any points $x_0 < \cdots < x_n$ in I, the collocation matrix $(\phi_j(x_i))_{i,j=0}^n$ is TP. If Φ is TP and $\sum_{i=0}^n \phi_i = 1$ (so that its collocation matrix is stochastic), we say that Φ is a NTP basis. Totally positive transformations have a variation diminishing property, defined as follows : if $\mathbf{T}$ is a totally positive matrix and $\mathbf{v}$ is any vector for which $\mathbf{Tv}$ is defined, then $S^-(\mathbf{Tv}) \leqslant S^-(\mathbf{v})$ (see [9]), where $S^-(\mathbf{v})$ denotes the number of strict sign changes in the (real) sequence of elements of the vector $\mathbf{v}$. Similarly, for a real-valued function f on an interval I we define $S^-(f)$ to be the number of sign changes of f, that is

$$S^-(f) = \sup S^-(f(x_0), \ldots, f(x_m))$$

where the supremum is taken over all increasing sequences $(x_0, \ldots, x_m)$ in I for all m. Thus from this and the variation diminishing property we have

$$S^-(\mathcal{B}_n(f; x)) \leqslant S^-(f(0), f([1]/[n]), \ldots, f(1)) \leqslant S^-(f).$$

This, together with the fact that generalized Bernstein polynomials reproduce linear functions, implies that when the function f is monotonic so is $\mathcal{B}_n(f; x)$

and when it is convex so is $\mathcal{B}_n(f;x)$ for any $0 < q \leqslant 1$. Consequently, the operator $\mathcal{B}_n$ preserves the shape of the function f on $[0,1]$, for any $0 < q \leqslant 1$.

Let us define the parametric curve $\mathsf{P}(t)$ by

$$\mathsf{P}(t) = (\mathsf{p}_1(t), \mathsf{p}_2(t)) = \sum_{i=0}^{n} \mathsf{b}_i B_i^n(t), \quad 0 \leqslant t \leqslant 1, \tag{2.2}$$

where $\mathsf{b}_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 0, \ldots, n$. We will write $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$ to denote the polygonal arc which joins up the points $\mathsf{b}_i = (x_i, y_i)$, $i = 0, \ldots, n$, using piecewise linear interpolation. Since the generalized Bernstein basis is a normalized totally positive basis for P_n and $0 < q \leqslant 1$, $\mathsf{P}(t)$ is a convex combination of the points $\mathsf{b}_0, \ldots, \mathsf{b}_n$. Thus $\mathsf{P}(t)$ must lie in the convex hull of the control points for all $0 \leqslant t \leqslant 1$. Another consequence is the variation diminishing property.

Theorem 2.1 *The number of times any straight line l crosses the curve $\mathsf{P}(t)$ defined by* (2.2) *is no more than the number of times it crosses $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$.*

This is indeed true for any NTP basis Φ, and is proved in [9]. It is obvious, on comparing the number of sign changes of $l = ax + by + c$ with that of P in (2.2), that we have

$$\begin{aligned} S^-(a\mathsf{p}_1 + b\mathsf{p}_2 + c) &= S^-\left(\sum (ax_i + by_i + c) B_i^n(t)\right) \\ &\leqslant S^-(ax_0 + by_0 + c, \ldots, ax_n + by_n + c), \end{aligned}$$

which gives the desired result.

It follows from this that if the polygonal arc $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$ is monotonic in a given direction then so is the curve P. Moreover, if $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$ is convex, then any straight line crosses it at most twice. Hence the curve P crosses any line at most twice which implies that P is convex. Thus the shape of the curve (2.2) mimics the shape of the control polygon $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$.

Since both $\Psi = (B_0^n(x), \ldots, B_n^n(x))$ and the power basis $\Phi = (1, x, \ldots, x^n)$ form a basis for the space of polynomials P_n, we may find the transformation matrix $\mathbf{M}$ such that $\Phi^{\mathrm{T}} = \mathbf{M}\Psi^{\mathrm{T}}$. Since $\sum_{k=0}^{n-j} B_k^{n-j}(x) = 1$ we have

$$x^j = \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix} x^{j+k} \prod_{t=0}^{n-j-k-1} (1 - q^t x).$$

On shifting the limits of the sum and the product above, using (1.4) and writing $\begin{bmatrix} n-j \\ k-j \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} / \begin{bmatrix} n \\ j \end{bmatrix}$, we obtain

$$x^j = \sum_{k=j}^{n} \frac{\begin{bmatrix} k \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} B_k^n(x), \quad j = 0, \ldots, n. \tag{2.3}$$

The matrix $\mathbf{M}$ has the entries

$$m_{j,k} = \frac{\begin{bmatrix} k \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} = \frac{[n-j]!\,[k]!}{[k-j]!\,[n]!}.$$

We may write $\mathbf{M} = \mathbf{ATB}$ such that $\mathbf{A}$ is a diagonal matrix with $a_{j,j} = \frac{[n-j]!}{[n]!}$, $\mathbf{B}$ is a diagonal matrix with $b_{k,k} = [k]!$ and $\mathbf{T}$ is a Toeplitz matrix with $t_{j,k} = 1/([k-j]!)$. Obviously the matrices $\mathbf{A}$ and $\mathbf{B}$ are TP for any $q > 0$. With a little work on the matrix $\mathbf{T}$ we can verify that it can be written as a product of 1-banded matrices such that $\mathbf{T} = \mathbf{T}^{(1)}\mathbf{T}^{(2)} \cdots \mathbf{T}^{(n)}$ and the elements of each factor are

$$t_{j,k}^{(i)} = \begin{cases} 1, & j = k, \\ \frac{q^{k-i}}{[k]}, & j = k-1,\ k \geqslant i. \end{cases}$$

Thus $\mathbf{T}$ is TP for any $q > 0$. Since the product of TP matrices is a TP matrix we conclude that $\mathbf{M}$ is a TP matrix.

We also invert the matrix $\mathbf{M}$ to obtain corresponding coefficients in $\Psi^{\mathrm{T}} = \mathbf{M}^{-1}\Phi^{\mathrm{T}}$. In a similar way, using (1.4), we have

$$B_j^n(x) = \sum_{k=j}^{n} (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} x^k. \tag{2.4}$$

Thus the inverse of $\mathbf{M}$ has elements

$$m_{j,k}^{-1} = (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}.$$

Let us take $\phi_i = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, 1, \ldots, n$. In the case of $\Psi^{\mathrm{T}} = \mathbf{M}\Phi^{\mathrm{T}}$, the elements of $\mathbf{M}$ satisfy

$$m_{i,j} = \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\binom{n}{j}} (1-q)^{j-i} S(n-1-i, j-i),$$

where $S(n,j)$ is the sum of $\binom{n}{j}$ possible products of j distinct factors chosen from the set $\{[1], [2], \ldots, [n]\}$. Note that $S(n,j)$ satisfies the following recurrence relation

$$S(n,j) = S(n-1,j) + [n]S(n-1,j-1),$$

as can easily be verified from its generating function

$$(1+x)(1+[2]x)\cdots(1+[n]x) = \sum_{j=0}^{n} S(n,j)x^j.$$

Since any polynomial curve can be expressed in terms of both bases,

$$\mathsf{P}(t) = \sum_{i=0}^{n} \mathsf{b}_i B_i^n(t) = \sum_{i=0}^{n} \mathsf{p}_i \phi_i(t),$$

the transformation matrix $\mathbf{M}$ also provides the relationship between the control points of P,

$$(\mathsf{b}_0, \ldots, \mathsf{b}_n)^{\mathrm{T}} = \mathbf{M}^{\mathrm{T}} (\mathsf{p}_0, \ldots, \mathsf{p}_n)^{\mathrm{T}}.$$

It can be shown that the matrix $\mathbf{M}$ obtained above is TP for any $0 < q \leqslant 1$ and $\mathbf{M}^{\mathrm{T}}$ is stochastic. Moreover it can be written as a product of 1-banded matrices as follows :

$$\mathbf{M} = \mathbf{A}^{n,q}\, \mathbf{T}^{1,q} \ldots \mathbf{T}^{n-1,q}\, (\mathbf{A}^{n,1})^{-1},$$

where $\mathbf{A}^{n,q}$ is the diagonal matrix with elements $a^{n,q}_{j,j} = \left[{n \atop j}\right]$ and $\mathbf{T}^{k,q}$ is the 1-banded matrix such that

$$t^{k,q}_{i,j} = \begin{cases} 1, & i = j, \\ 1 - q^{n-k}, & i = j-1, 0 \leqslant i < k \leqslant n-1, \\ 0, & \text{otherwise.} \end{cases}$$

This class of matrices is of particular interest in geometric design, when applying a corner cutting algorithm which is defined by a 1-banded TP stochastic matrix. It is shown in [8] that a matrix which is nonsingular, TP and stochastic can be written as a product of 1-banded matrices of the same type that describes a corner cutting algorithm. It is also shown in [3], by using this technique to obtain the Bézier polygon, that the Bernstein basis has optimal shape preserving properties among all NTP bases for P_n.

3. Degree elevation and reduction

One may wish to increase the flexibility of a given curve, using the technique of degree elevation. A degree elevation algorithm calculates a new set of control points by choosing a convex combination of the old set of control points which retains the old end points. For this purpose the identities

$$(1 - q^{n-j}t)B^n_j(t) = \frac{[n+1-j]}{[n+1]}\, B^{n+1}_j(t) \tag{3.1}$$

and

$$q^{n-j}t\, B^n_j(t) = \left(1 - \frac{[n-j]}{[n+1]}\right) B^{n+1}_{j+1}(t) \tag{3.2}$$

prove useful. These follow immediately from (2.1).

Theorem 3.1 *Let* $\mathsf{P}(t) = \sum_{j=0}^n \mathsf{b}_j B^n_j(t),\ \ 0 \leqslant t \leqslant 1.$ *Then*

$$\mathsf{P}(t) = \sum_{j=0}^{n+r} \mathsf{b}^r_j B^{n+r}_j(t), \tag{3.3}$$

where, for $n \geqslant 3$ and $i = 0, 1, \ldots, n+r$,

$$\mathsf{b}^r_i = \sum_{j=0}^{n} q^{(i-j)(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{\left[{r \atop i-j}\right]}{\left[{n+r \atop i}\right]}\, \mathsf{b}_j. \tag{3.4}$$

Proof This can be proved by induction on r, as follows. We write the given curve as $\mathsf{P}(t) = (1 - q^{n-j}t)\mathsf{P}(t) + q^{n-j}t\mathsf{P}(t)$ and apply the following recursive algorithm on degree elevated points

$$\mathsf{b}^r_i = \left(1 - \frac{[n+r-i]}{[n+r]}\right) \mathsf{b}^{r-1}_{i-1} + \frac{[n+r-i]}{[n+r]}\, \mathsf{b}^{r-1}_i \qquad \begin{cases} r = 1, 2, \ldots \\ i = 0, 1, \ldots, n+r \end{cases} \tag{3.5}$$

where $\mathsf{b}_i^0 = \mathsf{b}_i$. On writing

$$\left(1 - \frac{[n+r-i]}{[n+r]}\right) \Big/ \begin{bmatrix} n+r-1 \\ i-1 \end{bmatrix} = \frac{q^{n+r-i}}{\left[{n+r \atop i} \right]} \quad \text{and} \quad \frac{[n+r-i]}{[n+r]\left[{n+r-1 \atop i} \right]} = \frac{1}{\left[{n+r \atop i} \right]},$$

we can simplify the right side of (3.5) to give

$$\mathsf{b}_i^r = \sum_{j=0}^{n} q^{(i-j)(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{\left(q^{r+j-i}\left[{r-1 \atop i-j-1} \right] + \left[{r-1 \atop i-j} \right]\right)}{\left[{n+r \atop i} \right]} \, \mathsf{b}_j.$$

The q-binomial coefficients in the numerator of the latter expression are combined using (1.2) to give $\left[{r \atop i-j} \right]$. This verifies (3.4), and thus (3.3) holds. ■

When q is replaced by 1 above, we obtain the well known degree elevation process for Bézier curves. (See [5, 11, 6].) We observe from (3.5) that each new point is obtained by a convex combination of the two previous points. This suggests the following. Let $\mathbf{b}$ denote the vector such that $\mathbf{b}^{\mathrm{T}} = [\mathsf{b}_0, \ldots, \mathsf{b}_n]$, where the elements are the control vertices defined above. We also define $\mathbf{b}^r$ as the vector whose elements are the control vertices $\mathbf{b}_i^r$, $i = 0, 1, \ldots, n+r$, generated by repeating the degree elevation process r times.

Theorem 3.2 *Define the curve*

$$E^r \mathsf{P}(t) = \sum_{i=0}^{n+r} \mathsf{b}_i^r B_i^{n+r}(t)$$

obtained by r times degree elevation. Then, for $0 < q \leqslant 1$, the number of times the curve $E^r\mathsf{P}$ crosses any straight line l is bounded by the number of times the polygon $p(\mathsf{b}_0^r, \ldots, \mathsf{b}_{n+r}^r)$ crosses l.

Proof Let $\mathbf{T}^{r,n}$ be the transformation matrix such that $\mathbf{b}^r = \mathbf{T}^{r,n}\mathbf{b}$. It is enough to show that $\mathbf{T}^{r,n}$ is a TP matrix. Once again we use induction on r to prove that $\mathbf{T}^{r,n}$ is a product of r 1-banded positive matrices with the elements

$$T_{i,j}^{r,n} = \frac{q^{(i-j)(n-j)}\left[{n \atop j} \right]\left[{r \atop i-j} \right]}{\left[{n+r \atop i} \right]}. \tag{3.6}$$

Thus $T_{i,j}^{r,n}$ is zero unless $0 \leqslant i-j \leqslant r$. We note that the elements $T_{i,j}^{r,n}$ are the coefficients which appear in (3.4). Now, the result holds for $r = 0$ since $\mathbf{T}^{0,n}$ is simply the $(n+1)\times(n+1)$ identity matrix. Let $\mathbf{B}^{(r)}$ denote the $(n+r+1)\times(n+r)$ 1-banded positive matrix such that

$$B_{i,j}^{(r)} \;=\; q^{(i-j)(n+r-i)} \frac{\left[{n+r-1 \atop j} \right]}{\left[{n+r \atop i} \right]}, \quad \text{for } 0 \leqslant i-j \leqslant 1. \tag{3.7}$$

Then $\mathbf{T}^{r,n} = \mathbf{B}^{(r)}\mathbf{B}^{(r-1)}\cdots\mathbf{B}^{(1)}$. Let $\mathbf{V} = \mathbf{B}^{(r+1)}\,\mathbf{T}^{r,n}$. Explicitly this yields

$$V_{i,j} = \sum_{k=0}^{n+r+1} B_{i,k}^{(r+1)}\, T_{k,j}^{r,n}.$$

We see from (3.7) that $B^{(r+1)}_{i,k}$ is nonzero only for $k = i-1$ and $k = i$. Thus

$$V_{i,j} = B^{(r+1)}_{i,i-1}\, T^{r,n}_{i-1,j} + B^{(r+1)}_{i,i}\, T^{r,n}_{i,j}.$$

Hence, with $r+1$ in (3.7) and (3.6), we obtain

$$V_{i,j} = q^{(n+r-i+1)+(i-1-j)(n-j)} \frac{\left[{n \atop j}\right]\left[{r \atop i-1-j}\right]}{\left[{n+r+1 \atop i}\right]} + q^{(i-j)(n-j)} \frac{\left[{n \atop j}\right]\left[{r \atop i-j}\right]}{\left[{n+r+1 \atop i}\right]}$$

and hence

$$V_{i,j} = q^{(i-j)(n-j)} \frac{\left[{n \atop j}\right]}{\left[{n+r+1 \atop i}\right]} \left(q^{r-i+j+1} \begin{bmatrix} r \\ i-1-j \end{bmatrix} + \begin{bmatrix} r \\ i-j \end{bmatrix} \right).$$

Using the identity (1.2) we obtain

$$V_{i,j} = q^{(i-j)(n-j)} \frac{\left[{n \atop j}\right]\left[{r+1 \atop i-j}\right]}{\left[{n+r+1 \atop i}\right]} = T^{r+1,n}_{i,j},$$

thus completing the proof. ■

Thus the degree elevation process for the curve with the q-Bernstein basis is variation diminishing. This has the following consequences. If the control polygon $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$ is monotonic in the y direction, so is the degree elevated polygon $p(\mathsf{b}^r_0, \ldots, \mathsf{b}^r_{n+r})$. If the control polygon $p(\mathsf{b}_0, \ldots, \mathsf{b}_n)$ is convex, so is the degree elevated polygon $p(\mathsf{b}^r_0, \ldots, \mathsf{b}^r_{n+r})$.

The inverse process of degree elevation, which is called degree reduction, aims to represent a given curve of degree n by one of degree $n-1$. In general, exact degree reduction is not possible. We require the q-difference form of the Bernstein polynomials,

$$\mathcal{B}_n(f;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0\, x^r,$$

where $\Delta^r f_i = \Delta^{r-1} f_{i+1} - q^{r-1}\Delta^{r-1} f_i$, and an induction argument shows that

$$\Delta^r f_i = \sum_{k=0}^{r} (-1)^k q^{k(k-1)/2} \begin{bmatrix} r \\ k \end{bmatrix} f_{r+i-k}. \tag{3.8}$$

We deduce that a q-Bézier curve of degree n with control points $\mathsf{b}_0, \ldots, \mathsf{b}_n$ has a degree $n-1$ representation if and only if $\Delta^n \mathsf{b}_0 = \mathbf{0}$. Thus, from (3.8) we have

$$\Delta^n \mathsf{b}_0 = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} \mathsf{b}_{n-i} = \mathbf{0}.$$

In this case, in order to find the new points $\tilde{\mathsf{b}}_0, \ldots, \tilde{\mathsf{b}}_{n-1}$ for the q-Bézier representation of degree $n-1$ we use the degree elevation formulas (3.1) and (3.2) so that

$$\sum_{i=0}^{n} \mathsf{b}_i B^n_i(t) = \sum_{i=0}^{n-1} \tilde{\mathsf{b}}_i \left(\frac{[n-i]}{[n]} B^n_i(t) + \left(1 - \frac{[n-1-i]}{[n]}\right) B^n_{i+1}(t) \right).$$

On comparing the coefficients of the basis functions $B_i^n(t)$, we obtain

$$\mathsf{b}_i = \frac{[n-i]}{[n]}\tilde{\mathsf{b}}_i + \left(1 - \frac{[n-i]}{[n]}\right)\tilde{\mathsf{b}}_{i-1}, \quad i = 0, 1, \ldots, n-1,$$

from which we obtain

$$\tilde{\mathsf{b}}_i = \frac{[n]}{[n-i]}\mathsf{b}_i - \left(\frac{[n]}{[n-i]} - 1\right)\tilde{\mathsf{b}}_{i-1}, \quad i = 0, 1, \ldots, n-1. \tag{3.9}$$

This approximation is from the left of the control polygon, taking $\tilde{\mathsf{b}}_0 = \mathsf{b}_0$. When i is replaced by $n-i$ in (3.9) we have an approximation from the right side, with $\tilde{\mathsf{b}}_{n-1} = \mathsf{b}_n$,

$$\tilde{\mathsf{b}}_{n-i-1} = \frac{[n]}{[n]-[i]}\mathsf{b}_{n-i} - \frac{[i]}{[n]-[i]}\tilde{\mathsf{b}}_{n-i}, \quad i = 0, 1, \ldots, n-1. \tag{3.10}$$

It is well known that when $q = 1$, as the number of degree elevated points $r \to \infty$ the degree elevated polygon $p(\mathsf{b}_0, \ldots, \mathsf{b}_{n+r})$ tends to the original curve $\mathsf{P}(t)$ defined in the Theorem 3.1. with the rate $O(\frac{1}{n})$, see [4, 16, 7].

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