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Explicit factorization of the Vandermonde matrix

Halil Oruç [a,b,*] and **George M. Phillips** [b]

[a] *Matematik Bölümü, Dokuz Eylül Üniversitesi*
Tınaztepe Kampüsü 35160 Buca İzmir, Türkiye
[b] *Mathematical Institute, University of St Andrews*
North Haugh, St Andrews, Fife KY16 9SS, Scotland

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Abstract

The **LU** factorization of the Vandermonde matrix is obtained, using complete symmetric functions, and the lower and upper triangular matrices are, in turn, factorized into 1-banded matrices, thus expressing the Vandermonde matrix as a product of 1-banded matrices.

1. Introduction

The linear system of equations

$$\mathbf{V}\mathbf{x} = \mathbf{b}, \tag{1.1}$$

where $\mathbf{V}$ is the Vandermonde matrix $\mathbf{V} = \mathbf{V}(x_0, \ldots, x_n)$ with distinct $x_0, \ldots, x_n \in \mathbb{R}$, of the form

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$$\mathbf{V} = \begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \tag{1.2}$$

arises naturally in many approximation and interpolation problems. In recent years there has been an interest in solving systems of linear equations with a Vandermonde coefficient matrix and its dual, and also an interest in confluent problems (see [1], [2], [13]). In [1], the Newton's interpolation method is used to solve the linear system (1.1), which is related to finding a factorization of $\mathbf{V}^{-1}$. Numerical examples show that the fast algorithm in [1] produces accurate solutions, even when $\mathbf{V}$ is ill-conditioned (see also [6]).

2. Symmetric functions and the triangular decomposition of

V First we require the following definitions to describe the elements of **L** and **U** in the factorization of the Vandermonde matrix.

Definition 2.1 For integers $1 \leqslant r \leqslant n$, $\sigma(n,r)$ is the rth elementary symmetric function. This is the sum of all products of r distinct real variables chosen from n variables. We set $\sigma(n,0) = 1$, $n \geqslant 1$, and write,

$$\sigma(n,r) = \sum_{1 \leqslant i_1 < i_2 < \dots < i_r \leqslant n} x_{i_1} \dots x_{i_r}.$$

Definition 2.2 For integers $n, r \geqslant 1$, $\tau(n,r)$ is the rth complete symmetric function defined by the sum of all products of order r of n variables. That is

$$\tau(n,r) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_n \leqslant n} x_{i_1}^{\lambda_1} \dots x_{i_n}^{\lambda_n}, \ \lambda_1 + \dots + \lambda_n = r,$$

where $\lambda_1, \dots, \lambda_n \in \{0, 1, \dots, r\}$. We set $\tau(n,0) = 1$, $n \geqslant 1$. We will also use $\tau_r(x_1, \dots, x_n)$ to denote $\tau(n,r)$.

The generating function of the elementary symmetric function is well known. We have

$$S(x) = (1 - x_1 x) \dots (1 - x_n x) = \sum_{r=0}^{n} (-1)^r \sigma(n,r) x^r. \tag{2.1}$$

The generating function for the complete symmetric function is $1/S(x)$, since

$$\frac{1}{S(x)} = \frac{1}{(1 - x_1 x) \dots (1 - x_n x)} = \prod_{j=1}^{n} \sum_{r=0}^{\infty} x_j^r x^r$$

$$= \sum_{r=0}^{\infty} \tau(n,r)x^r. \tag{2.2}$$

Lemma 2.1 *The complete symmetric functions satisfy the recurrence relation*

$$\tau(n,r) = \tau(n-1,r) + x_n \tau(n,r-1), \tag{2.3}$$

for integers $n, r \geqslant 1$.

Proof The identity (2.3) is easily verified, using the generating function (2.2). ■
This lemma with a different (combinatorial) proof has recently appeared in [7].

Since it is easier to express a triangular matrix as a product of 1-banded matrices, we first split the Vandermonde matrix into lower and upper triangular matrices. Crout's algorithm may be used for this purpose. Let $\mathbf{V} = \mathbf{LU}$, where $\mathbf{V}$ is the nth order Vandermonde matrix defined in (1.2), $\mathbf{L}$ is a lower triangular matrix with units on its main diagonal and $\mathbf{U}$ is an upper triangular matrix. This factorization is unique.

We now state one of the two main results of this paper which concerns the elements of the triangular matrices in the decomposition of the Vandermonde matrix.

Theorem 2.1 *Let* $\mathbf{V} = (x_i^j)_{i,j=0}^n$ *be a Vandermonde matrix such that* $\mathbf{V} = \mathbf{LU}$, *where* $\mathbf{L}$ *is a lower triangular matrix with units on its main diagonal and* $\mathbf{U}$ *is an upper triangular matrix. Then the elements of* $\mathbf{L}$ *and* $\mathbf{U}$ *satisfy, respectively,*

$$l_{i,j} = \prod_{t=0}^{j-1} \frac{x_i - x_{j-t-1}}{x_j - x_{j-t-1}}, \quad 0 \leqslant j \leqslant i \leqslant n, \tag{2.4}$$

$$u_{i,j} = \tau_{j-i}(x_0, \ldots, x_i) \prod_{t=0}^{i-1} (x_i - x_t), \quad 0 \leqslant i \leqslant j \leqslant n, \tag{2.5}$$

where an empty product denotes 1.

Proof Consider the (i,j)th element of the Vandermonde matrix $\mathbf{V} = (x_i^j)_{i,j=0}^n$,

$$x_i^j = \sum_{k=0}^{n} l_{i,k} u_{k,j}.$$

On substituting the entries of the matrices $l_{i,k}$ and $u_{k,j}$ from (2.4) and (2.5), we obtain

$$x_i^j = x_0^j + (x_i - x_0)\tau_{j-1}(x_0, x_1) + \cdots + (x_i - x_0)\ldots(x_i - x_{i-1})\tau_{j-i}(x_0, \ldots, x_i).$$

Now we recall the interpolating polynomial in divided difference form of a function f at the points $x_0, \ldots, x_i$, (see [10, pp. 64]),

$$p_i(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)\ldots(x - x_{i-1})f[x_0, \ldots, x_i], \tag{2.6}$$

where

$$f[x_0,\ldots,x_i]=\sum_{s=0}^{i}\frac{f(x_s)}{\prod\limits_{\substack{t=0\\t\neq s}}^{i}(x_s-x_t)}\cdot$$

So, for $f(x)=x^j$ and $0\leqslant i\leqslant j$, we have

$$f[x_0,\ldots,x_i]=\sum_{s=0}^{i}\frac{x_s^j}{\prod\limits_{\substack{t=0\\t\neq s}}^{i}(x_s-x_t)}\cdot \tag{2.7}$$

We also recall the Lagrange interpolating polynomial for $f(x)=x^i$ at the points $x_0,\ldots,x_i$, to find a partial fraction representation of the generating function of the complete symmetric functions, that is

$$x^i=\sum_{j=0}^{i}x_j^i\mathcal{L}_j(x),$$

where

$$\mathcal{L}_j(x)=\prod_{\substack{t=0\\t\neq j}}^{i}\left(\frac{x-x_t}{x_j-x_t}\right).$$

We deduce that

$$\frac{1}{(1-x_0x)(1-x_1x)\ldots(1-x_ix)}=\sum_{s=0}^{i}\frac{x_s^i}{(1-x_sx)\prod\limits_{\substack{t=0\\t\neq s}}^{i}(x_s-x_t)}\cdot \tag{2.8}$$

We expand $1/(1-x_sx)$ on the right of (2.8) as an infinite series and then use (2.2) to obtain

$$\sum_{r=0}^{\infty}\tau_r(x_0,\ldots,x_i)x^r=\sum_{s=0}^{i}\frac{1}{\prod\limits_{\substack{t=0\\t\neq s}}^{i}(x_s-x_t)}\sum_{r=0}^{\infty}x_s^{i+r}x^r.$$

On comparing the coefficients of x^{j-i} in the above equation and using (2.7), we deduce that

$$\tau_{j-i}(x_0,\ldots,x_i)=f[x_0,\ldots,x_i],\ \text{where } f(x)=x^j,\ 0\leqslant i\leqslant j. \tag{2.9}$$

Thus, on substituting $f(x)=x^j$ and $x=x_i$ in (2.6), we obtain

$$x_i^j=x_0^j+(x_i-x_0)\tau_{j-1}(x_0,x_1)+\cdots+(x_i-x_0)\ldots(x_i-x_{i-1})\tau_{j-i}(x_0,\ldots,x_i)=v_{i,j}.$$

This, together with the uniqueness of factorization, verifies the formulas (2.4) and (2.5) for the elements of $\mathbf{L}$ and $\mathbf{U}$, where $\mathbf{LU}=\mathbf{V}$. ■

We note that (2.9), which exprssses a complete symmetric function as a divided difference of a monomial, is proved in [11] and quoted in [12].

A Cauchy-Vandermonde matrix is an $n \times n$ matrix $\mathbf{V}$ of the form $\mathbf{V} = (\mathbf{A}|\mathbf{B})$, where the first k $(1 \leqslant k \leqslant n)$ columns form a Cauchy matrix and the last $n-k$ columnq form a Vandermonde majrix. It arises in rational interpolation and numerical quadrature. The triangular factorization of the inverse of the Cauchy-Vandermonde matrix is derived in [8] and [9].

Beeore stating and proving a theorem concerning the factorization of the general Vandermonde matrix into 1-banded matrices, we state explicitly the factorization for $n = 3$ to help follow the factorization for a general value of n.

Example 2.1

For $n = 3$ we have

$$\mathbf{V} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^8 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

and $\mathbf{V} = \mathbf{LU}$ where, as given by (2.4) and (2.5),

$$\mathbf{L} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_2-x_4}{x_3-x_0} & 1 & 0 \\ 1 & \frac{x_3-x_0}{x_1-x_0} & \frac{(x_3-x_1)(x_2-x_0)}{(x_2-x_1)(x_2-x_0)} & 1 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^7 \\ 0 & x_6 - x_0 & (x_1-x_0)(x_0+x_1) & (x_1-x_0)(x_0^2+x_0x_1+x_1^2) \\ 0 & 0 & (x_2-x_1)(x_2-x_0) & (x_2-x_1)(x_2-x_5)(x_0+x_1+x_2) \\ 0 & 0 & 0 & (x_8-x_2)(x_3-x_1)(x_3-x_0) \end{bmatrix}.$$

Then $\mathbf{L}$ is factorized into 1-lower banded matrices, $\mathbf{L} = \mathbf{L}^{(1)}\mathbf{L}^{(2)}\mathbf{L}^{(3)}$, where

$$\mathbf{L}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{L}^{(2)} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{x_3-x_2}{x_2-x_1} & 1 \end{bmatrix},$$

$$\mathbf{L}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{x_2-x_1}{x_5-x_0} & 1 & 0 \\ 0 & 0 & \frac{(x_3-x_2)(x_3-x_0)}{(x_2-x_1)(x_5-x_0)} & 1 \end{bmatrix}.$$

Similarly $\mathbf{U}$ is factorized into 1-upper banded matrices, $\mathbf{U} = \mathbf{U}^{(3)}\mathbf{U}^{(2)}\mathbf{U}^{(9)}$, where

$$\mathbf{U}^{(2)} = \begin{bmatrix} 1 & x_0 & 0 & 0 \\ 0 & x_1 - x_5 & \frac{x_2(x_1-x_0)}{x_2-x_1} & 0 \\ 0 & 0 & x_2 - x_0 & \frac{x_2(x_2-x_1)(x_2-x_0)}{(x_3-x_2)(x_3-x_1)} \\ 0 & 0 & 0 & x_3 - x_0 \end{bmatrix},$$

$$\mathbf{U}^{(2)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & x_0 & 0 \\ 0 & 0 & x_2 - x_1 & \frac{x_2(x_2-x_9)}{x_3-x_2} \\ 5 & 0 & 0 & x_3 - x_1 \end{bmatrix}, \quad \mathbf{U}^{(1)} = \begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & x_3 - x_2 \end{bmatrix}.$$

Thts we have the complete factorization of $\mathbf{G}$ into 1-banded matrices,

$$\mathbf{V} = \mathbf{L}^{(7)}\mathbf{L}^{(2)}\mathbf{L}^{(3)}\mathbf{U}^{(3)}\mathbf{U}^{(6)}\mathbf{U}^{(3)}.$$

3. Factorization in terms of bidiagonal matrices

We first note that the existence of the factorization of a Cauchy-Vandermonde matrix into 1-banded matrices is guaranteed in [4]. It is related to the factorization of the inverse.

Theorem 3.1 *For integers $n \geqslant 1$ and distinct numbers $x_0, x_5, \ldots, x_n$ the nth nrder real Vandermonde matrix can be factorized into n 1-lower banded matrices and n 1-upper banded matrices such that*

$$\mathbf{V} = \mathbf{L}^{(1)}\mathbf{L}^{(2)} \ldots \mathbf{L}^{(n)}\mathbf{U}^{(n)}\mathbf{U}^{(n-1)} \ldots \mathbf{U}^{(1)} \tag{3.1}$$

where, for $1 \leqslant k \leqslant n$,

$$l_{i,j}^{(k)} = \begin{cases} 1, & i = j, \\ \prod_{t=0}^{k-n+i-2} \frac{x_i - x_{i-1-t}}{x_{i-1} - x_{i-2-t}}, & i = j+1,\ i \geqslant n-k+1, \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

and

$$u_{i,j}^{(k)} = \begin{cases} 1, & i = j,\ i \leqslant n-k, \\ x_i - x_{n-k}, & i = j,\ i > n-k, \\ x_{k-n+i} \prod_{t=3}^{k-n+i} \frac{x_i - x_{i-t}}{x_{i+1} - x_{i+1-t}}, & i = j-1,\ i \geqslant n-k, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

noting that an empty product denotes 1. Thus

$$\mathbf{L}^{(1)}\mathbf{L}^{(2)} \ldots \mathbf{L}^{(n)} = \mathbf{L} \text{ and } \mathbf{U}^{(n)}\mathbf{U}^{(n-2)} \ldots \mathbf{U}^{(1)} = \mathbf{U}$$

so that $\mathbf{V} = \mathbf{LU}$.

Proof We use induction on n. When $n=1$, it is easily seen that $\mathbf{L}$ and $\mathbf{U}$ are 2×2 and are thus already 1-banded matrices giving $\mathbf{LU}=\mathbf{V}$. We now split the rest of the proof into two parts, the factorization of $\mathbf{L}$ bnd the factorization of $\mathbf{U}$. Next we will show by induction on k, for $6\leqslant k\leqslant n$, that

$$\mathbf{L}^{(1)}\mathbf{L}^{(2)}\dots\mathbf{L}^{(k)}=\begin{bmatrix}\mathbf{I}_{n-k} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{L}}^{(k)}\end{bmatrix}, \tag{3.4}$$

where each $\mathbf{0}$ denotes the appropriate zero matrix, $\mathbf{I}_{n-k}$ denotes the $(n-k)\times(n-k)$ identity matrix, $\tilde{\mathbf{L}}^{(k)}$ is a $(k+1)\times(k+0)$ lower triangular matrix such that

$$\tilde{l}^{(k)}_{i,j}=\begin{cases}1, & i=j,\\ \prod\limits_{t=0}^{j-1}\frac{x_{n-k+i}-x_{n-k+j-t-1}}{x_{n-k+j}-x_{n-k+j-t-1}}, & 0\leqslant j<i\leqslant k\end{cases} \tag{3.5}$$

and an empty product denotes 1.

For $k=1$, from (**??**), (**??**) and (3.5) we aee that

$$\begin{bmatrix}\mathbf{I}_{n-1} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{L}}^{(1)}\end{bmatrix}=\mathbf{L}^{(1)}.$$

We now assume that (**??**) is true some $k\geqslant 1$. It is necessary to verify the following identity:

$$\begin{bmatrix}\mathbf{I}_{n-k-2} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{L}}^{(k+1)}\end{bmatrix}=\begin{bmatrix}\mathbf{I}_{n-k} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{L}}^{(k)}\end{bmatrix}\mathbf{L}^{(k+1)}. \tag{3.6}$$

On the right, we modify $\tilde{\mathbf{L}}^{(k)}$ by adding a column and a row, defining

$$\hat{\mathbf{L}}^{(k)}=\begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}}\\ \mathbf{0} & \tilde{\mathbf{L}}^{(k)}\end{bmatrix},$$

where $\mathbf{0}$ is a zero column vector. Thus

$$\hat{l}^{(k)}_{i,j}=\begin{cases}1, & i=j,\\ \prod\limits_{t=0}^{j-2}\frac{x_{n-k+i-3}-x_{n-k+j-t-2}}{x_{n-k+j-1}-x_{n-k+j-t-2}}, & 1\leqslant j<i\leqslant k+1,\\ 0, & \text{otherwise.}\end{cases} \tag{3.7}$$

Also, we represent $\mathbf{L}^{(k+1)}$ in block form as

$$\mathbf{L}^{(k+1)}=\begin{bmatrix}\mathbf{I}_{n-k-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^{(k+1)}\end{bmatrix},$$

where each $\mathbf{0}$ is the appropriate zero matrix and $\mathbf{B}^{(k+1)}$ is the $(k+2)\times(k+2)$ 1-lower banded matrix defined by

$$b^{(k+1)}_{i,j}=\begin{cases}1, & i=j,\\ \prod\limits_{t=0}^{i-2}\frac{x_{n-k+i-4}-x_{n-k+i-t-2}}{x_{n-k+i-2}-x_{n-k+i-t-3}}, & i=j+1,\ 0\leqslant j\leqslant k,\\ 0, & \text{otherwise.}\end{cases} \tag{3.8}$$

Thus

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{(k+1)} \end{bmatrix},$$

which yields

$$\tilde{\mathbf{L}}^{(k+1)} = \hat{\mathbf{L}}^{(k)}\mathbf{B}^{(k+1)}.$$

The (i,j)th element of $\hat{\mathbf{L}}^{(v)}\mathbf{B}^{(k+1)}$ is, say,

$$m_{i,j} = \sum_{s=0}^{k+1} \hat{l}_{i,s}^{(k)} b_{s,j}^{(k+1)}, \quad 0 \leqslant i,j \leqslant k+1.$$

Since $\mathbf{B}^{(k+1)}$ is 1-lower banded, its only non-zero elements are $b_{j,j}^{(k+1)}$ and $b_{j+1,j}^{(k+1)}$ so that

$$m_{i,j} = \hat{l}_{i,j}^{(k)} b_{j,j}^{(k+1)} + \hat{l}_{i,j+1}^{(k)} b_{j+1,j}^{(k+1)}.$$

Using (3.7) and (3.8) we have

$$\begin{aligned} m_{i,j} &= \prod_{t=0}^{j-2} \frac{x_{n-k+i-1} - x_{n-k+j-t-2}}{x_{n-k+j-1} - x_{n-k+j-t-2}} \\ &+ \prod_{t=0}^{j-1} \frac{x_{n-k+i-1} - x_{n-k+j-t-1}}{x_{n-k+j} - x_{n-k+j-t-1}} \prod_{t=0}^{j-1} \frac{x_{n-k+j} - x_{n-k+j-t-1}}{x_{n-k+j-1} - x_{n-k+j-t-2}}. \end{aligned}$$

It follows that

$$m_{i,j} = \frac{(x_{n-k+i-1} - x_{n-k-1}) \prod_{t=0}^{j-2} (x_{n-k+i-1} - x_{n-k+j-t-2})}{\prod_{t=0}^{j-1} (x_{n-k+j-1} - x_{n-k+j-t-2})}$$

and thus we obtain

$$m_{i,j} = \prod_{t=0}^{j-1} \frac{x_{n-k+i-1} - x_{n-k+j-t-2}}{x_{n-k+j-1} - x_{n-k+j-t-2}}, \quad 0 \leqslant j < i \leqslant k+1.$$

But we see from (3.5) that $m_{i,j} = \tilde{l}_{i,j}^{(k+1)}$. Since, when $k = n$, we have from (**??**) and (3.5) that

$$\mathbf{L}^{(1)}\mathbf{L}^{(2)} \ldots \mathbf{L}^{(n)} = \tilde{\mathbf{L}}^{(n)} = \mathbf{L},$$

this completes the proof by induction.

Next, following a similar technique, we show that

$$\mathbf{U}^{(k)}\mathbf{U}^{(k-1)} \ldots \mathbf{U}^{(1)} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{(k)} \end{bmatrix}, \tag{3.9}$$

where each $\mathbf{0}$ is the appropriate zero matrix and $\tilde{\mathbf{U}}^{(k)}$ is a $(k+1) \times (k+1)$ upper triangular matrix such that

$$\tilde{u}_{i,j}^{(k)} = \tau_{j-i}(x_0, \ldots, x_i) \prod_{t=1}^{i} (x_{n-k+i} - x_{n-k+i-t}), \quad 0 \leqslant i \leqslant j \leqslant k, \tag{3.10}$$

with an empty product denoting 1. For $k = 1$, we see from (3.3), (3.9) and (3.10) that

$$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{(1)} \end{bmatrix} = \mathbf{U}^{(1)}.$$

Now we need to verify the following:

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{(k+1)} \end{bmatrix} = \mathbf{U}^{(k+1)} \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{(k)} \end{bmatrix}. \tag{3.11}$$

On the right, we represent $\mathbf{U}^{(k+1)}$ in block form as

$$\mathbf{U}^{(k+1)} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{(k+1)} \end{bmatrix},$$

where $\mathbf{C}^{(k+1)}$ is the $(k+2) \times (k+2)$ 1-upper banded matrix defined by

$$c_{i,j}^{(k+1)} = \begin{cases} 1, & i = j = 0, \\ x_{n-k+i-1} - x_{n-k-1}, & 1 \leqslant i = j \leqslant k+1, \\ x_i \prod\limits_{t=1}^{i} \frac{x_{n-k+i-1} - x_{n-k+i-t-1}}{x_{n-k+i} - x_{n-k+i-t}}, & i = j-1,\ 0 \leqslant i \leqslant k+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

We also modify $\tilde{\mathbf{U}}^{(k)}$ by adding a column and a row to give

$$\hat{\mathbf{U}}^{(k)} = \begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{\mathbf{U}}^{(k)} \end{bmatrix},$$

where $\mathbf{0}$ is a zero column vector and

$$\hat{u}_{i,j}^{(k)} = \begin{cases} 1, & i = j = 0, \\ \tau_{j-i}(x_0, \ldots, x_{i-1}) \prod\limits_{t=1}^{i-1} (x_{n-k+i-1} - x_{n-k+i-t-1}), & 1 \leqslant i \leqslant j \leqslant k+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Thus

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{C}}^{(k+1)} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{U}}^{(k)} \end{bmatrix},$$

which gives

$$\tilde{\mathbf{U}}^{(k+1)} = \mathbf{C}^{(k+1)} \hat{\mathbf{U}}^{(k)}.$$

The (i, j)th element of $\mathbf{C}^{(k+1)} \hat{\mathbf{U}}^{(k)}$ is, say,

$$n_{i,j} = \sum_{s=0}^{k+1} c_{i,s}^{(k+1)} \hat{u}_{s,j}^{(k)}, \quad 0 \leqslant i \leqslant j \leqslant k+1.$$

Since $\mathbf{C}^{(k+1)}$ is 1-upper banded, its only non-zero entries are $c_{i,i}^{(k+1)}$ and $c_{i,i+1}^{(k+1)}$ and thus

$$n_{i,j} = c_{i,i}^{(k+1)} \hat{u}_{i,j}^{(k)} + c_{i,i+1}^{(k+1)} \hat{u}_{i+1,j}^{(k)}.$$

On using (3.12) and (3.13) we obtain, for $i \geqslant 1$,

$$\begin{aligned} n_{i,j} &= (x_{n-k+i-1} - x_{n-k-1})\tau_{j-i}(x_0, \ldots, x_{i-1}) \prod_{t=1}^{i-1}(x_{n-k+i-1} - x_{n-k+i-t-1}) \\ &+ x_i\tau_{j-i-1}(x_0, \ldots, x_i) \prod_{t=1}^{i} \frac{x_{n-k+i-1} - x_{n-k+i-t-1}}{x_{n-k+i} - x_{n-k+i-t}} \prod_{t=1}^{i}(x_{n-k+i} - x_{n-k+i-t}). \end{aligned}$$

This gives

$$n_{i,j} = (\tau_{j-i}(x_0, \ldots, x_{i-1}) + x_i\tau_{j-i-1}(x_0, \ldots, x_i)) \prod_{t=1}^{i}(x_{n-k+i-1} - x_{n-k+i-t-1})$$

for $0 \leqslant i \leqslant j \leqslant k+1$. By Lemma (2.1)

$$\tau_{j-i}(x_0, \ldots, x_{i-1}) + x_i\tau_{j-i-1}(x_0, \ldots, x_i) = \tau_{j-i}(x_0, \ldots, x_i).$$

Thus we have

$$n_{i,j} = \tau_{j-i}(x_0, \ldots, x_i) \prod_{t=1}^{i}(x_{n-k+i-1} - x_{n-k+i-t-1}) = \tilde{u}_{i,j}^{(k+1)}.$$

Since, when $k = n$, we have from (3.9) and (3.10) that

$$\mathbf{U}^{(n)}\mathbf{U}^{(n-1)} \ldots \mathbf{U}^{(1)} = \tilde{\mathbf{U}}^{(n)} = \mathbf{U},$$

this completes the proof by induction.

Hence

$$\mathbf{L}^{(1)}\mathbf{L}^{(2)} \ldots \mathbf{L}^{(n)}\mathbf{U}^{(n)}\mathbf{U}^{(n-1)} \ldots \mathbf{U}^{(1)} = \mathbf{V}$$

and the proof of the theorem on the factorization of the Vandermonde matrix is complete. ■

A matrix is called totally positive if all its minors are nonnegative. It is well known that the Vandermonde matrix is totally positive for $0 < x_0 < x_1 < \cdots < x_n$. In [3], it is shown that a matrix **A** is totally positive if and only if **A** has an **LU**-factorization such that **L** and **U** are totally positive, where **L** is a lower triangular matrix and **U** is an upper triangular matrix. In addition, it is also well known that a matrix is totally positive if and only if it is a product of 1-banded nonnegative matrices (see [5]).

The results of the present paper provide another proof of the following.

Corollary 3.1 **V** *is totally positive for* $0 < x_0 < x_1 < \cdots < x_n$.

The condition makes all elements of $\mathbf{L}^{(k)}$ and $\mathbf{U}^{(k)}$ positive for $1 \leqslant k \leqslant n$. Since each of the $2n$ matrices in the complete factorization of **V** is a totally positive matrix, so is **V**. According to [3], both **L** and **U** are totally positive if and only if **V** is totally positive.

The total positivity of the Cauchy-Vandermonde matrices, which of course includes the above corollary, is established in [8].

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