INTERPOLATION METHODS APPLIED IN
COMPUTER AIDED ENGINEERING

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ABSTRACT

Computer Aided Engineering is nowadays a wide domain which includes: experimental methods, CAD/CAM, data processing programs, dedicated numerical methods (FEM, CFD, BEM), etc. As it can be noticed, the main instrument of investigation is the computer, and the computer methods naturally include numerical methods in the same way how computer hardware requires an operating system.

In engineering, general numerical methods are used to solve different specific problems. In particular, interpolation methods are extensively applied in the models of the different phenomena where experimental data must be used in computer studies where expressions of those data are required.

The paper emphasizes the importance of the numerical analysis in mechanical engineering software applications, being provided a systematic presentation of the methods and techniques of numerical analysis and interpolation of the functions. Basically, there are many types of approximating functions. Thus, any analytical expression may be expressed as an approximating function, the most common types being: polynomials, trigonometric and exponential functions. Special attention is dedicated to polynomials which are the oldest and simplest methods of approximation.

In many problems in engineering and science, the data consist of sets of discrete points, being required approximating functions which must have the following properties:

- the approximating function should be easy to determine;
- it should be easy to evaluate;
- it should be easy to differentiate;
- it should be easy to integrate.

It can be noticed that polynomials satisfy all four these properties.

Keywords: Interpolation, Applications, Engineering

1. INTRODUCTION

Numerical methods are today a common use instrument which is, so to say, just a click away. Several years ago researchers used to analyze the engineering problems and to create their own software applications which used, most of the time, original numerical methods libraries.

Nowadays, there are many software applications which offer facilities regarding the numerical analysis of the data, in a certain, well defined context. Despite this popular approach where colourful graphics are easily created, there are profound aspects which are not always well understood and applied in an engineering problem. This is why, a researcher must know the mathematical background of the problems and to evaluate the applicability and the limits of the ready-to-use software instruments.

A particular and important aspect in the numerical methods subject is the approximation of the different values, operation designated as interpolation, which is employed in most of the branches of the science, such
This paper is dedicated to the interpolation methods, which is the subject of the PhD Thesis of one of the authors.

2. THEORETICAL BACKGROUND
Numerical analysis uses special methods for functions’ approximation. In approximation problems for functions with one or more real variable, we have to choose also one approximation criteria.

The most useful approximation methods are: approximation using interpolation, approximation with spline functions, finite elements.

In this paper is a short presentation of polynomial interpolation and of the approximation using spline functions.

2.1 Polynomial interpolation
The interpolation can be performed by approximating the unknown law of variation with an analytical function. The problem of function approximation arise when we know only the numerical values of function or when the function is very complicated.

Generally, a problem solved by interpolation approximation can be formulated as: let \( f : A \to B \), \( A, B \) non null sets, we suppose known the value \( y_p \) of function \( f \) on given data points \( x_p \in A \). That means \( f(x_p) = y_p \), \( p = 0,1,...,n \). We must find a real or complex interpolation function \( F : A' \to B' \), \( (A \subseteq A', B \subseteq B') \), which has to satisfy the following conditions:

\[
F(x_p) = y_p, \quad p = 0,...,n
\]  

(1)

This kind of general problem of interpolation should have a unique solution, no solution or infinity of solutions.

Geometrically that means we have to find a curve by equation \( y = F(x) \) which should pass through the points \( M_p(x_p, y_p) \), \( p = 0,...,n \).

In other way, instead of \( F \) function we consider a specific fit value to \( f(\bar{x}) \), where \( x_p < \bar{x} < x_{p+1} \). In practical problems we take in consideration that \( f(\bar{x}) \approx F(\bar{x}) \) and then we evaluate the error term defined as \( f(\bar{x}) - F(\bar{x}) \).

The function used for approximation belongs to one class of functions. The most useful class of functions are: monoms, trigonometrical and exponential functions. Linear combination of monoms produce \( n \) order polynoms. The most useful is the polynomial approximation.

Considering the conditions of interpolation it results a \( n \) equation system with \( n \) unknowns. It is desirable that the system includes only linear equations.

The general form of such system is:

\[
\sum_{i=1}^{n} a_i x^{i-1} = y_k, \quad k = 1,...,n
\]  

(2)

If \( n \) is large, the solving of system (2) is slightly more difficult. The solution of such system consists in the coefficients of polynomial approximation.

To solve the system in a readily way, we have to choose functions on interval \([a,b]\) which contains the distinct points (nodes) \( x_0, x_1,..., x_p \). These points are part of linear space of finite dimension of \( C[a,b] \). If on the interval \([a,b]\), we consider one polynom, then the approximation will be a global approximation. The theoretical base of polynomial approximation is the Weierstrass theorem. This theorem shows that any continuous function can be approximated with accuracy on an interval, using a polynomial function.
The interpolation polynomial function is unique for a function on any given interval. The most known methods for polynomial interpolation are: Lagrange, Newton, Hermite, Birkhoff polynomial interpolation, trigonometrical and rational interpolation.

2.2 Lagrange interpolation method

For \( p + 1 \) arbitrary support points \((x_p, y_p), p = 0, \ldots, n, x_p \neq x_r, \) for \( p \neq r \) there is a unique polynomial \( P(x) = f_p, p = 0, \ldots, n \)

\[
P(x) = \sum_{p=0}^{n} f_p L_p(x) = \sum_{p=0}^{n} f_p \prod_{r \neq p} \frac{x - x_r}{x_p - x_r}
\]

(3)

The above interpolation formula shows that the coefficients of the \( P \) function depend linearly on the support ordinates \( f_p \). Lagrange’s formula may, however, be useful in some situations in which many interpolation problems are to be solved for the same support abscissa \( x_p, p = 0, \ldots, n \), but using different sets of support ordinates \( f_p, p = 0, \ldots, n \).

2.3 The convergence of interpolating polynomials

If a continuous function \( f \) is defined on an interval \([a, b]\), and if interpolating polynomials \( P \) of higher degree are constructed for \( f \), the natural expectation is that these polynomials will uniformly converge to \( f \) on \([a, b]\).

That means that we expect that the expression

\[
\max_{a \leq x \leq b} |f(x) - P(x)|
\]

will converge to 0 when \( n \to \infty \). Some well known examples for a real domain were given by Runge C. in 1901. This is the function \( f(x) = \frac{1}{1 + x^2}, x \in [-5, 5] \) and Lagrange polynomials relative at \( x_i = -5 + \frac{10}{m}, i = 0, \ldots, m \)

If interpolating polynomials \( P \) are constructed for this function using equally spaced nodes in \([-5, 5]\), it is found that the sequence \( \|f - P\|_\infty \) is not bounded.

For any prescribed system of nodes

\[
a \leq x_0 < x_1 \ldots < x_n \leq b, n \geq 0
\]

(4),

there exists a continuous function \( f \) on \([a, b]\) for which the interpolating polynomials for \( f \) using these nodes fail to converge uniformly to \( f \).

This matter is a little more subtle than it seems at the first glance, so we have to present the following theorem.

**THEOREM**

If \( f \) is a continuous function on \([a, b]\), there is a system of nodes, as defined in (4), for which the interpolation polynomials \( P \) of \( f \) satisfy

\[
\lim_{n \to \infty} \|f - P\|_\infty = 0
\]

in these nodes.

This theorem results by joining the Weierstrass and Chebyshev powerful theorems.

2.4 The Weierstrass approximation theorem

If \( f \) is continuous on \([a, b]\) and if \( \varepsilon > 0 \), then there is a polynomial \( P \) satisfying

\[
|f(x) - P(x)| \leq \varepsilon \quad \text{on the interval } [a, b].
\]

Let \( L_p (p \geq 1) \) be a sequence of positive linear operators defined on \( C[a, b] \) and with values in the same space. If

\[
\|L_p f - f\|_\infty \to 0
\]

(5)
for the three functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, then relation (5) is true for all $f \in C[a, b]$.

### 2.5 Interpolation using spline functions

Spline functions lead to smooth interpolating curves which are less likely to exhibit the large oscillations characteristic of high-degree polynomials. High-degree polynomials would obviously pass through all the data points themselves, but they can oscillate wildly between data points due to round-off errors and overshoot. Spline approximation is used in both graphical applications and numerical methods. In such cases, lower-degree polynomials can be fit to subsets of the data points. An alternate approach is to use lower-degree polynomials which must to connect to each other in all the data points and to require these polynomials to be consistent with each other in some sense. This type of polynomial is called a spline function.

Spline function can be of any degree. Linear splines are simply straight line segments connecting each pair of data points. Linear splines are first-order approximating polynomials. Quadratic splines are second-order approximating polynomials. The slopes of the quadratic splines can be constrained to be continuous at each data point, but the curvatures are still discontinuous.

A cubic spline is a third-degree polynomial connecting each pair of data points. The slopes and curvatures of the cubic splines can be constrained to be continuous at each data point. Higher-degree splines can be defined in a similar manner.

Let us consider the discrete $x$ space and the following indexing rule: there are $n+1$ total points, $x_i$, $i = 1, 2, ..., n + 1$, $n$ intervals, and $n - 1$ interior grid points, $x_i$, $i = 1, 2, ..., n$. A cubic spline must be fit to each interval. Thus,$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, i = 1, 2, ..., n$ (6) defines the cubic spline in interval $I_i, x_i \leq x \leq x_{i+1}, i = 1, 2, ..., n$. Since each cubic spline has four coefficients and there are $n$ cubic splines, there are $4 \cdot n$ unknown coefficients. It results that $4 \cdot n$ boundary conditions or constraints must be available.

When all of the conditions given above are assembled, $4 \cdot n$ linear algebraic equations are obtained for the $4 \cdot n$ unknown spline coefficients $d_i, c_i, b_i$, and $a_i, i = 1, 2, ..., n$. This system of equations can be solved by Gauss elimination.

For the cubic equation (6), it is obvious that the second derivative within each interval, $f''_i(x)$, is a linear function of $x$. The first-order Lagrange polynomial for the second derivative $f''_i(x)$ in interval $i$, $x_i \leq x \leq x_{i+1}, i = 1, 2, ..., n$, is given by:

$$f''_i(x) = \frac{x-x_{i+1}}{x_i-x_{i+1}} f''_i + \frac{x-x_i}{x_{i+1}-x_i} f''_{i+1}$$ (7)

Integrating equation (6) it results:

$$f'_i(x) = \frac{x^2}{2} \frac{2}{x_i-x_{i+1}} f''_i + \frac{2}{2} \frac{x^2}{x_{i+1}-x_i} f''_{i+1} + E$$ (8)

$$f'_i(x) = \frac{x^3}{6} \frac{2}{x_i-x_{i+1}} f''_i + \frac{6}{2} \frac{x^2}{x_{i+1}-x_i} f''_{i+1} + E \cdot x + F$$ (9)

Evaluating equation (8) at $x_i$ and $x_{i+1}$ and combining the results to eliminate the constants of integration $E$ and $F$ it results:
Equation (10) is the desired cubic spline for increment $i$ expressed in terms of the two unknown second derivatives $f_i''(x)$ and $f_i'''(x)$.

An expression for the second derivatives at interior grid points, $f_i''(x)$, $i = 2, 3, \ldots, n$ can be obtained by setting $f_i''(x_i) = f_i''(x_{i+1})$. An expression for $f_i''(x_i)$ can be obtained by differentiating equation (10).

Considering $f_1'' = f_{n+1}'' = 0$, it results the most commonly employed approach.

3. Applications in engineering

3.1 General Ideas Regarding the Use of the Interpolation in the Numerical Analysis

There are certain cases when approximation solutions are useful within the numerical methods field itself. Some of these cases are presented in the next sections.

3.1.1 Approximation of the Initial Solution for Iterative Methods

Let us consider a theoretical problem where the formulation of the problem depends on some parameters and an iterative solution is possible. It is well known that the final solution depends on the initial approximation in terms of accuracy and of computer runtime.

Figure 1: Approximation is used to evaluate an initial solution which is close to the final solution.
If some solutions can be found using an initial solution which is not accurate, the next initial solutions can be evaluated by the use of an interpolation method. In this way the computer time decreases and the accuracy is increased. Figure 1 illustrates this case.

3.1.2 Calculus of the Integrals

Let us consider that a set of points are known, that is $P_j(x_j, y_j), j = 1, N$. Usually these points result from experimental studies. There must be computed the integral of the function approximated by this set of points.

$$\sum_{j=1}^{N} \frac{(Y_j + Y_{j+1}) \cdot (X_{j+1} - X_j)}{2 \cdot \text{Area } A_j}$$

$$\sum_{j=1}^{N} \int_{a_j}^{b_j} f_j(x) \, dx$$

Figure 2 – Two ways to compute the integral of a function defined by a set of points
A first method to compute the integral is to use the summation of the areas of the trapezoids defined by two consecutive points. This is a coarse approximation, but if the points are close one to the other the accuracy may be considered satisfactory.

The second method has two stages. The first one is to approximate the set of points using one of the previously mentioned interpolation methods, and it results a set of equations like (6). The second stage is to analytically integrate the functions and to calculate the result. In this case the source of errors is in the interpolation stage.

A similar approach can be used to compute the derivative of the functions, either using the initial points, or using the appropriate interpolation functions.

The error for each of the methods may be evaluated for a certain dataset. Basically, it is necessary to use as many methods exist, in order to have several results which, ideally, must be close one to the other.

3.2 General Ideas Regarding the Interpolation in the Finite Element Method

Finite Element Method is nowadays a classic computing method, so only some of the main ideas will be presented in this section. The principle of the method is to replace the continuum of the domain with some subdomains designated as finite elements. These elements “communicate” one with the other by the use of the nodes which define the geometrical shape of each element, being considered some nodal parameters as constraints, deflections, etc. To represent in an accurate way the phenomenon inside the element, there are considered these nodal parameters together with some internal fields for the variables which are significant in a given context; for instance there can be considered fields of displacements, slopes, curvatures, stresses, strains, etc. To have a precise value for a given field in a certain point, there are used interpolation functions which consider the coordinates of the point where the values must be computed. In a classic approach:

\[ p(x) = N_1(x) \cdot p_1 + N_2(x) \cdot p_2 + \ldots + N_j(x) \cdot p_j + \ldots + N_n(x) \cdot p_n = \sum_{j=1}^n N_j(x) \cdot p_j \quad (11), \]

where:
- \( p(x) \) is the polynomial approximation function;
- \( p_j \) is the value of the previous function in the “j” node;
- \( N_j(x) \) is the interpolation function associated to the “j” node (its value is 1 in “j” and 0 for all the rest of the nodes).

The Lagrange interpolation functions do not consider constraints for the derivatives of the interpolation functions. Many physical problems require compatibility conditions which mean that the nodal unknowns consist in the value of the physical parameter together with its derivatives. In this case there must be used Hermite interpolation functions.

As an example, the interpolation functions are:

\[ N_i = \frac{\prod_{j=1 \atop j \neq i}^n (x-x_j)}{\prod_{j=1 \atop j \neq i}^n (x_i-x_j)} \quad (12). \]

For the bidimensional and the tridimensional problems, the interpolation functions are:

\[ N_{ij} = N_i^m(x) \cdot N_j^n(y) \quad (13), \]

and

\[ N_{ijk} = N_i^m(x) \cdot N_j^n(y) \cdot N_k^p(z) \quad (14), \]

where \( m \), \( n \) and \( p \) are the intervals on the \( x \), \( y \) and \( z \) directions; other significance of these values is the maximum degree of the interpolation functions; other meaning is the number of parentheses in the according equation (12).
3.3 Trigonometric Approximations in CAE

Let us consider a simple and relevant example. In solid mechanics, the tangential stresses inside a rectangle cross-section beam loaded with a torque can be computed with the expressions:

\[
\begin{align*}
\tau_{\text{max}} &= \frac{M_r}{\alpha \cdot h \cdot b^2} \\
\tau_{\text{max max}} &= \frac{M_r}{\alpha' \cdot h \cdot b^2}
\end{align*}
\]  

(15)

The angle of relative rotation between two sections is

\[
\Delta \varphi = \frac{M_r \cdot l}{\beta \cdot h \cdot b^3 \cdot G}
\]  

(16).

Coefficients $\alpha$, $\alpha'$ and $\beta$ are functions which depend on the $\frac{h}{b} > 1$ parameter and they can be usually found in tables.

![Diagram of stresses in a rectangular section of a beam loaded with a torque](image_url)

**Figure 3:** Stresses in a rectangular section of a beam loaded with a torque

For precise CAE models the values can be computed using some trigonometric approximations, solutions conceived in the theory of elasticity.
\[ \chi = \frac{1}{3} \left\{ 1 - \frac{192}{\pi^5} \frac{b}{h} \sum_{n=1,3,5}^{\infty} \frac{1}{n^5} \sinh \left( \frac{n \pi h}{2b} \right) \right\} \]

\[ \tau_{yx} = \frac{M_y}{\chi \cdot h \cdot b^5} \cdot \frac{8}{\pi^2} \sum_{n=1,3,5}^{\infty} \left\{ (-1)^{n-1} \frac{n^2}{n^2} \frac{\sinh \left( \frac{n \pi h}{2b} \right)}{\cosh \left( \frac{n \pi h}{2b} \right)} \cdot \cos \left( \frac{n \pi y}{b} \right) \right\} \]  

(17)

These expressions are convergent and a fair accuracy can be reached for the first 5 terms. The use of a large number of terms leads to unexpected errors specific to the numeric programming.

3.4 Applications of the Moving Average in Experimental Data Processing

As mentioned before, important sources of the points considered in interpolation problems are the experiments. For instance, a thorough experimental and numerical study was carried out regarding the state of strains and stresses in the block cylinder of a military naval engine, in running conditions.

The experimental data were used to calibrate the numerical FEM model, so the experimental data was checked, double checked and over checked in order to offer a high degree of confidence. Several states of the running conditions were studied and several millions of experimental values were acquired. In this way there was verified the repeatability of the experimental data for all the measurement points.

The slight differences are normal for the real conditions where the complete and precise repeatability cannot be reached.

As it can be noticed, there are several local oscillations of the measured data, being necessary a method to smooth the curves.
The most facile method to smooth the curves was to use Excel in order to add a trend line which might present the relevant variation of the phenomenon. Excel also offers a whole set of facilities, such as:
- import of the data from several formats: text, Visual FoxPro tables, comma separated values, etc;
- graphics of the data presented in the sheet;
- record, store and run a macro (a set of consecutive commands), in this way being possible to draw a graph by using a single click.

Analysing the variation of the moving average trend line for several numbers of terms to be considered in the calculus of the average, it results a set of concluding remarks, such as:
- being several points in the graph with inherent local oscillations, it is necessary a large number of terms in the average, in order to smooth the curve;
- most of the experimental values have the same final trend line for a fair large number of terms;
- the average was automatically computed using the values which follow the current value, so it was an obvious shift to the right of the final curve.

A customized trend line was created by considering the current value as the middle point of the interval of values included in the average. The resulting curve was compared with the default moving average trend line and there was noticed their close similarity of the shapes.

### 3.5 Approximation Using CAD Facilities

Nowadays, the interpolation facilities are at a click away in most of the applications, including the CAD software applications.
SolidEdge offers a set of powerful facilities regarding the interpolation, using b-spline functions. This approximation can be employed for both 2D and 3D curves. Further on, these curves can be used to define 3D surfaces. Thus, there can be created non-analytic surfaces when a b-spline curve is swept or extruded, or when one constructs lofted, swept or BlueSurf feature using the b-spline curves.

Conversion of an analytically defined geometry into a curve is a basic facility in several CAD applications.

Figure 7 – B-spline interpolation for 2D curves in SolidEdge
As it can be noticed in the previous figure, the spline approximation takes into consideration the so-called “Fit tolerance” parameter. The values considered for this parameter are important for the overall accuracy of the approximation.

Sometimes the definition of the spline curve requires the slope of the curves at the both ends, this operation being done by defining the two tangents.

### 3.6 Polynomial Approximation of Different Technical Data

Experimental mechanics has several techniques to obtain information regarding a real phenomenon. Strain gage technology is one of the most accurate methods employed in the experimental studies. Regarding the parasitic effects, there were developed experimental research methodologies to eliminate, to self-compensate, to compensate and to correct this kind of influences.
The variation of the temperature during an experiment is one of the most important parasitic effects to be considered in the strain gage technology.

In order to correct the values of the measured strains, responsible strain gage manufacturers offer for each package some information regarding the corrections to be performed. For instance, the corrections of the so-called ‘Thermal Output’ are done by considering the following law of variation:

\[
\varepsilon_{\text{correction}}^{\Delta T} = A_0 + A_1 \cdot T^1 + A_2 \cdot T^2 + A_3 \cdot T^3 + A_4 \cdot T^4
\]  

(18)

After the computing of this “Thermal Output” correction, the value is used to adjust the measured strains. Equations (19) are used for a three-direction rosette, in this case a so-called “rectangle” rosette.

\[
\begin{align*}
\varepsilon_{\text{corrected}}^{10} &= \varepsilon_{10} - \varepsilon_{\text{correction}}^{\Delta T} \\
\varepsilon_{\text{corrected}}^{145} &= \varepsilon_{145} - \varepsilon_{\text{correction}}^{\Delta T} \\
\varepsilon_{\text{corrected}}^{90} &= \varepsilon_{90} - \varepsilon_{\text{correction}}^{\Delta T}
\end{align*}
\]  

(19)

To conclude, the previous figure presents a comparison between the calculated values and the values resulted from the law of variation offered by the manufacturer of the strain gages, which is obviously an interpolation function. The two curves are close one to the other, so the analyst may consider equation (18) in the experimental data processing algorithm/software.

4. CONCLUSION

Marine engineering is another field where approximation is used in different operations. For instance, in the latest Model Course 7.04 for Officer in Charge of an Engineering Watch, page 236, it is recommended that the interpolation can be used in that given context. It can be noticed that linear approximation is the most facile way to manual compute the interpolated values, without a fair accuracy. A computer program would offer more accurate results in less time. Anyway, interpolation is also used as a background instrument in the onboard software applications used by a maritime officer.
Aspects regarding the interpolation appear in many problems in computing, engineering, economics, etc. Taking into account the actual trends and the wide range of problems considered, there must be emphasized the following ideas:

- there are many facilities in several software applications which propose different solutions to the interpolation problem;
- most of the applications are not flexible enough to customize a general solution for a given particular problem;
- there are several background methods to solve the interpolation problems (moving average, spline, polynomials, Lagrange/Hermite in FEM, trigonometric approximations, etc.);
- the mathematical aspects are paramount in order to make the most proper decisions regarding the interpolation problems, decisions which can lead to poor accuracy, wrong technical solutions, failures and losses;
- the user of the software applications which offer such advanced facilities should not be an operator who can get simple colourful graphics, but an intelligent analyst who takes into consideration several aspects before choosing a certain solution.

Example - Find the draughts after loading 250 tonnes in No. 1 hold.

<table>
<thead>
<tr>
<th>Initial draught forward 5.76 m</th>
<th>aft 6.38 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correction</td>
<td>+0.30</td>
</tr>
<tr>
<td>New draught</td>
<td>6.06 m</td>
</tr>
</tbody>
</table>

Notes:
1. **Interpolation can be used for intermediate draughts.**
2. **Reverse the sign of the corrections for discharged weights.**

Figure 10: Recommendation regarding the use of the interpolation for officers in charge of engineering watch.
Many other interesting aspects could be also included in the paper, but there were considered the most relevant ones. The presentation of the paper contains several other aspects.

5. ACKNOWLEDGEMENT

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