A Fortran 77 computer code for damped least-squares inversion of Slingram electromagnetic anomalies over thin tabular conductors

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Abstract

A FORTRAN 77 computer code is presented that permits the inversion of Slingram electromagnetic anomalies to an optimal conductor model. Damped least-squares inversion algorithm is used to estimate the anomalous body parameters, e.g. depth, dip and surface projection point of the target. Iteration progress is controlled by maximum relative error value and iteration continued until a tolerance value was satisfied, while the modification of Marquardt’s parameter is controlled by sum of the squared errors value. In order to form the Jacobian matrix, the partial derivatives of theoretical anomaly expression with respect to the parameters being optimised are calculated by numerical differentiation by using first-order forward finite differences.

A theoretical and two field anomalies are inserted to test the accuracy and applicability of the present inversion program. Inversion of the field data indicated that depth and the surface projection point parameters of the conductor are estimated correctly, however, considerable discrepancies appeared on the estimated dip angles. It is therefore concluded that the most important factor resulting in the misfit between observed and calculated data is due to the fact that the theory used for computing Slingram anomalies is valid for only thin conductors and this assumption might have caused incorrect dip estimates in the case of wide conductors.

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1. Introduction

Slingram (so-called Horizontal Loop Electromagnetic, HLEM) is a maximum coupled system which uses horizontal transmitter and receiver coil pairs kept at a fixed distance apart (usually 30, 60 or 90 m). The receiver system normally measures both in-phase and quadrature components of the secondary field as a percentage of the primary field intensity. The system is commonly used in exploring for conductive ore bodies and for groundwater exploration in fractured zones (Palacky et al., 1981; McNeill, 1990).

In electromagnetic methods, calculation of the response of a conductive target with arbitrary shape is more complicated than that in other geophysical methods. The structures for which an analytical solution can be obtained are rather limited, e.g. a thin perfectly conductive vertical or dipping dike, sphere and disk-shaped conductors (Grant and West, 1965). Duckworth et al. (1991) suggested a method for quantitative depth estimates by transforming Slingram anomalies to a form which is free from the effect of the coil separation, however, the interpretation of Slingram data is generally achieved using type curves or Argand diagrams (Lowrie and West, 1965; Nair et al., 1968; Parasnis, 1971;

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Nomenclature

\[ H(x) \] Slingram electromagnetic anomaly
\[ x \] distance of the point of observation (midpoint of the transmitter and receiver coils) from an arbitrary origin
\[ z \] depth to top of thin conductor
\[ \alpha \] dip of thin conductor (varies from 0° to 90°)
\[ X_0 \] surface projection point of thin conductor
\[ L \] distance between transmitter and receiver coils
\[ N \] number of observation points
\[ \text{RMS} \] root mean square error between observed data and calculated model response
\[ E \] sum of the squared errors
\[ R \] maximum relative error
\[ \lambda \] Marquardt’s parameter
\[ F \] vector of order \( N \) containing differences between observed and calculated data
\[ p_i \] initial estimates of the parameters
\[ \delta p \] error vector of order 3 containing the increments or decrements in the parameters
\[ A \] Jacobian matrix \( (N \times 3) \)
\[ A^T \] transpose matrix \( (3 \times N) \) of \( A \)
\[ I \] identity matrix \( (3 \times 3) \)
\[ B \] \( (A^T A) \) matrix \( (3 \times 3) \)
\[ G \] \( (A^T F) \) vector of order 3
\[ \Delta z \] amount of perturbation in depth \( (z) \) parameter
\[ \Delta \alpha \] amount of perturbation in dip \( (\alpha) \) parameter
\[ \Delta X_0 \] amount of perturbation in surface projection point \( (X_0) \) parameter

Fig. 1. Schematic illustration of geometrical relationship between coil system and perfectly conductive thin target. Parameters optimized are burial depth \((z)\), dip angle \((\alpha)\) and surface projection point \((X_0)\) of upper edge of conductor. \( L \) is distance between transmitter \((T_x)\) and receiver \((R_x)\) coils (redrawn from Duckworth and Krebes, 1995).

Hanneson and West, 1984). For this purpose, Ketola and Puranen (1967) offered a detailed model catalogue for the type curves for Slingram measurements.

As an adjunct to conventional analysis performed using type curves, we offer a FORTRAN 77 code named SLINV.FOR for the quantitative interpretation of Slingram measurements. The program computes the parameters of a dipping conductive dike type mineral deposit from Slingram measurements using the damped least-squares inversion algorithm suggested by Marquardt (1963). The parameters obtained are burial depth \((z)\), dip angle \((\alpha)\) measured from vertical and the surface projection point \((X_0)\) of the vein (Fig. 1).

Forward modeling theory presented by Wesley (1958) was used in the least-squares algorithm. Since the theory is based on the perfectly conductive thin conductors, Slingram anomalies are purely in-phase and do not have a quadrature component. The theory has also been widely used by various researchers (i.e. Grant and West, 1965; Ketola and Puranen, 1967; Duckworth et al., 1993). Duckworth and Krebes (1995) proposed the expression of normalized secondary field at the receiver coil as a percentage of the secondary field using the notation provided by Wesley (1958) as

\[
H(x) = 100 \left\{ L^3 \left[ \frac{\tan^{-1}(a/L)}{L^3} - \frac{\pi}{2} + \frac{a}{p^2 L^2} \right] + L^3 \left[ \frac{\pi}{2q} + \frac{\tan^{-1}(b/q)}{q^3} + \frac{b}{q^3 p^2} \right] + \frac{L^3(q^2 - 3L^2) \sin^2 x \left( \frac{\pi}{2q} + \frac{\tan^{-1}(b/q)}{q^3} + \frac{b}{q^3 p^2} \right)}{\pi} \right. \\
+ \frac{2L^3b \sin^2 x}{\pi q^3 p^4} [a \cos 2\alpha - c \cos 2\alpha] \\
+ \frac{L^3}{4\pi r \rho} \left[ a + b \cos 2\alpha - c \sin 2\alpha \right] \\
+ \frac{L^3 \sin x}{\pi p^4 r \rho} \left[ ((x_i + x_j) - L \cos x) \right. \\
\times (d \sin x + a \cos x)] \\
- \frac{L^3 \sin x}{\pi p^4 r \rho} \left[ ((x_i + x_j) - L \cos x) \right. \\
\times (d \sin x - a \cos x]] \\
+ \frac{L^3}{2\pi r \rho} \left[ (bd^2 - ac^2) \sin^2 x \\
+ abc \sin 2\alpha - ab(a + b) \cos^2 x \right],
\]  

(1)

where \( L \) is the coil separation and \( \alpha \) is the dip angle of the conductor (Fig. 1). Other notations used for
simplifying Eq. (1) are as follows:
\[
\begin{aligned}
x_t &= (x - X_0) \cos z + z \sin z + L_t \cos z, \\
x_i &= (x - X_0) \cos z + z \sin z - L_t \cos z, \\
z_t &= z \cos z - (x - X_0) \sin z - L_t \sin z, \\
z_i &= z \cos z - (x - x_0) \sin z + L \sin z,
\end{aligned}
\]
\[
\begin{aligned}
r_t &= \sqrt{x_t^2 + z_t^2}, \\
r_i &= \sqrt{x_i^2 + z_i^2}, \\
a &= 2 \sqrt{r_tr_i} \cos \left(\frac{\phi_t - \phi_i}{2}\right), \\
b &= 2 \sqrt{r_tr_i} \cos \left(\frac{\phi_t + \phi_i - 3\pi}{2}\right), \\
c &= 2 \sqrt{r_tr_i} \cos \left(\frac{\phi_t + \phi_i}{2}\right), \\
d &= 2 \sqrt{r_tr_i} \cos \left(\frac{\phi_t - \phi_i - 3\pi}{2}\right), \\
p &= \sqrt{L^2 + a^2}, \\
q &= \sqrt{(x_t + x_i)^2 + (L \sin z)^2},
\end{aligned}
\]
\[
\begin{aligned}
z_t < 0 &\quad \Rightarrow \phi_t = -\frac{\pi}{2} - \tan^{-1} \frac{x_t}{z_t}, \\
z_t > 0 &\quad \Rightarrow \phi_t = \frac{\pi}{2} - \tan^{-1} \frac{x_t}{z_t}, \\
z_i < 0 &\quad \Rightarrow \phi_i = -\frac{\pi}{2} - \tan^{-1} \frac{x_i}{z_i}, \\
z_i > 0 &\quad \Rightarrow \phi_i = \frac{\pi}{2} - \tan^{-1} \frac{x_i}{z_i},
\end{aligned}
\]
where \(x\) is the midpoint distance of transmitter and receiver coils. According to the forward solution theory of Wesley (1958) in Eq. (1), the dip angle of the conductor, \(z\), should be in the range of 0–90°. If the dip is in the other direction (e.g. less than 0°), the direction of the profile should be reversed for the inversion process.

2. Inversion method

In inverse modeling, a geometrical model is chosen with an initial guess of the body parameters and then the process is iteratively advanced until a satisfactory fit is obtained between observed and calculated anomalies. Several methods, such as the gradient method, ridge regression, the Gauss method and singular value decomposition have been used by other authors to determine target parameters automatically, especially for the inversion of potential field data arising from anomalous bodies with simple geometrical shapes (e.g. Whitehill, 1973; Atchuta et al., 1985; Venkata Raju, 2003, etc). The optimisation procedure modifies the initial body parameters iteratively to reach at a model which fits the observed data resulting in a best-fit model (Thanassoulas et al., 1987; Marobhe, 1989).

In geophysical inversion methods involving parameter optimisation, initial approximate values of model parameters are assumed and improvements in the values of these parameters are made in such a way that the calculated values \(F_{\text{cal}}\) of the best fitting model are close to observed data values \(F_{\text{obs}}\). The theoretical expression in Eq. (1) is nonlinear with respect to the parameters of the anomalous target, and therefore, the problem can be solved by the least-squares inversion method which can be formulated as follows:
\[
F = (F_{\text{obs}} - F_{\text{cal}}),
\]
where \(F\) is the vector containing differences of theoretical and observed data, \(N\) is the number of observation points, \(E\) is sum of the squared errors and RMS is the root mean square error. In order to minimize \(E\) using least squares, one can use the following generalised expression (Thanassoulas et al., 1987)
\[
A_{x_i} \Delta p = F,
\]
where \(\Delta p\) is the error vector containing the increments or decrements of the parameters in order to reduce \(E\) and \(A\) is the Jacobian matrix, each \((i,j)\) th element of which contains the partial derivative of the \(x_i\) th calculated data point with respect to the \(p_j\) th parameter \(\partial F_{\text{cal}}/\partial p_j\). Since \(A\) is not a square matrix, the transpose of matrix \(A\) is used to get the least-squares solution (Venkata Raju, 2003),
\[
\Delta p = [A^T A]^{-1} A^T F.
\]
Eq. (5) is known as Gauss–Newton solution which yields a rapid convergence for good initial estimates of the parameters. The possibility of singularity problem in the matrix \((A^T A)\), which may lead to a divergent solution, is overcome in Marquardt’s approach (Marquardt, 1963) by adding a constant \(\lambda\) to the principal diagonal elements of matrix \((A^T A)\). Then Eq. (5) becomes
\[
\Delta p = [A^T A + \lambda I]^{-1} A^T F,
\]
where \(\lambda\) is known as Marquardt’s parameter (or damping factor) and \(I\) is the unit diagonal matrix. The inversion progress given by Eq. (6) depends on the selection of \(\lambda\) value. In the present situation, an initial value of 0.01 for \(\lambda\) is chosen. If the inversion converges, \(\lambda\)
is reduced by dividing it by a constant factor of 10. If divergence occurred after one of the iterations, then \( \lambda \) was multiplied by the same factor until convergence resumes. Fig. 2 shows the modification of the Marquardt’s parameter, \( \lambda \), and the reduction of the RMS error value after each iteration.

In general, the inversion process is continued until a tolerance value is satisfied between observed and calculated data by controlling either \( E \) in Eq. (2) or the RMS values in Eq. (3) after each iteration. In the present study, the progress of the iteration is controlled by maximum relative error value \( R \); given by

\[
R = \frac{\Delta p_i}{p_i} = \frac{p_i - p_i'}{p_i},
\]

where \( p_i \) is the \( i \)th parameter to be optimised.

3. Calculation of partial derivatives

In order to constitute the Jacobian matrix, we need to have partial derivatives of the analytical expression of Slingram anomaly with respect to the parameters being estimated. Because the analytical solution of the simple geometrical model on which the inversion will be performed is already known, the expressions for the partial derivatives are usually obtained by analytical differentiation in the inversion process. In the present study, however, the partial derivatives with respect to depth \( z \), dip angle \( \alpha \) and the surface projection point of top of the perfectly conductive target \( X_0 \) were calculated by numerical differentiation since the analytical expression of the anomaly in Eq. (1) is quite complicated to differentiate analytically. In the numerical differentiation, the forward finite differences method was used. In general form of first order finite differences, the derivative of any function \( f(x) \) at a point \( x_i \) is given by

\[
\frac{\partial f(x)}{\partial x_i} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \quad (i = 1, \ldots, N),
\]

where \( \Delta x \) is the amount of perturbation of the function \( f(x) \). Using the notation given in Eq. (8), the partial derivatives of the expression in Eq. (1) with respect to the parameters \( z, \alpha \) and \( X_0 \) for each observation point can be calculated as

\[
\frac{\partial F_{cal}^X}{\partial z} = \frac{F_{cal}^X(z + \Delta z) - F_{cal}^X(z)}{\Delta z},
\]

\[
\frac{\partial F_{cal}^X}{\partial \alpha} = \frac{F_{cal}^X(\alpha + \Delta \alpha) - F_{cal}^X(\alpha)}{\Delta \alpha},
\]

\[
\frac{\partial F_{cal}^X}{\partial X_0} = \frac{F_{cal}^X(X_0 + \Delta X_0) - F_{cal}^X(X_0)}{\Delta X_0}.
\]

In computing partial derivatives for each iteration, the perturbation values of each parameter \( \Delta z, \Delta \alpha \) and \( \Delta X_0 \) are taken as 3% of the parameter values (e.g. \( \Delta z = 0.03z \), \( \Delta \alpha = 0.03 \alpha \) and \( \Delta X_0 = 0.03X_0 \)). The function values obtained from Eqs. (9) to (11) constitute the corresponding columns of the Jacobian matrix.
4. Description of the program

The main program SLINV.FOR contains 4 functions and 12 subroutines, each indicated in italics in Fig. 3. The program reads coil separation (L), initial estimates of the parameters (pi), a tolerance value and observed anomaly values \( F_{\text{obs}}^{i} \) with corresponding distances of observation points \( x_i \) from input data file SLINPUT.DAT and prints out the final iteration number (ITER), initial estimates and optimised parameters (pi), observed and calculated data \( (F_{\text{obs}}^{i} \text{ and } F_{\text{cal}}^{i}) \) with differences between the two, sum of the squared errors \( (E) \) and root mean square error (RMS) to the output data file named as SLOUTPUT.DAT after the inversion is completed.

Subroutine FVEC calculates the difference between observed and calculated data, sum of the squared error and root mean square error values. The calculated data \( F_{\text{cal}}^{i} \) is supplied by function FX which calculates the forward solution provided by Eq. (1). Functions FXA, FXB and FXC compute the partial derivatives of the

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Fig. 3. Flow chart of program SLINV.FOR. Names of functions and subroutines are given in italics. Program reads data from input data file SLINPUT.DAT and writes results to output data file named as SLOUTPUT.DAT. Iteration number is controlled by tolerance value supplied by user, whereas modification of \( \lambda \) value is controlled by sum of squared error value. See text for description of sub-programs.
analytical solution with respect to \(z\), \(a\) and \(X_0\) parameters respectively by using finite differences numerical differentiation as given by Eqs. (9)-(11) and supply the results to subroutine DERIV which forms the Jacobian matrix (\(A\)). Subroutine TRANS computes the transpose of the Jacobian matrix (\(A^T\)). Subroutine MATMULT performs matrix multiplication (\(A^T A\)) to produce matrix \(B\). Subroutine VECMUL computes (\(A^T F\)) matrix to give vector \(G\). Subroutine NORMAL performs the normalisation process of \(B\) and \(G\), and subroutine ADDLAM adds the \(\lambda\) value to the main diagonal elements of matrix \(B\). Subroutine CHOLESKY uses Cholesky’s method to solve the equation system (\(B \delta p = G\)) for \(\delta p\) correction values which are added to the parameters in subroutine ADDPAR after the unnormalisation process in subroutine UNNORM. Subroutine ERROR calculates the maximum relative error (\(R\)) given by Eq. (7). Then the program checks the \(R\) values for each parameter and makes a comparison between the maximum \(R\) value and the tolerance. If the inequality \(R < \text{Tolerance}\) occurs during any iteration, the iteration process is terminated and the results are stored into the output file. During the inversion procedure, Marquardt’s parameter value (\(\dot{\lambda}\)) is controlled by sum of the squared error value (\(E\)) in subroutine RELAM. Because the use of RMS value as an indicator of the fit between observed and calculated responses is more common in literature, the program also calculates RMS value although it is not actively used in the present inversion scheme.

5. Examples

5.1. Application to theoretical data

In order to test the efficiency and accuracy of the present inversion program, a theoretical anomaly is produced for a coil separation of \(L = 60\) m (200 ft) with the actual body parameters \(z = 20\) m, \(a = 45^\circ\) and \(X_0 = 150\) m using Eq. (1). Fig. 4 shows the theoretical data and its inversion result. The final solution obtained at the 7th iteration with an RMS error of 0.0028 shows satisfactory fit between theoretical and optimised curves.

5.2. Application to field data

Two Slingram anomalies obtained over a dipping graphitic shale from Northern Australia (Figs. 5A and 6A) were inverted in order to compare the results with those from Duckworth et al. (1991) who described the conductor as a perfectly conductive bed in a free space environment, because the quadrature component of the anomaly was almost featureless. Duckworth et al.
have shown that the dip of the conductor was towards west with a larger positive peak in the western side of the Slingram anomaly. In the present inversion scheme, however, we need to reverse the directions of the both profiles in Figs. 5A and 6A because Wesley’s forward solution theory suggests that the dip of the conductor should be in the east direction and hence dip angle of the conductor should be in the range of 0–90°.

Fig. 5A shows the in-phase component of the Slingram anomaly and its inversion results together with the geological cross section (Fig. 5B). The final solution was obtained after 16 iterations with RMS error of 4.68 and the parameters obtained are also given in Fig. 5A together with the parameters given by Duckworth et al. (1991) and drilling result for comparison. Fig. 6A shows another Slingram anomaly profile over the same conductor and its inversion results with the geological cross section (Fig. 6B). The parameters obtained from inversion are also given in Fig. 6A to compare those from Duckworth et al. (1991) and from drilling result. The estimated parameters were obtained again at the 16th iteration with an RMS error of 5.93.

6. Results and discussion

The method presented may provide a useful basis for the quantitative interpretation of Slingram anomalies. It is evident from the results given in Figs. 5 and 6 that the most reliable parameters provided by the method are estimates of depth (z) and surface projection point (X₀) of the conductor. The largest difference, however, between the results of Duckworth et al. (1991) and the present study was observed in dip estimates of the conductor.

Considering the assumptions of the theory used for calculating the theoretical Slingram anomalies, which also limit the application of the method, the factors resulting in the misfit between observed and the calculated data can be classified into three groups. In addition to the existence of geological and man-made noise on the field data, the misfit may also arise from the fact that the theoretical anomaly expression in Eq. (1) is valid only for

(1) thin conductors,
(2) perfectly conductive targets, and
(3) the conductors having infinite depth extent.
Because the targets detected by Slingram electromagnetic method are usually very good conductors, and because the depth extent of these conductors is usually greater than the coil separation, we can infer that factors 2 and 3 above do not have a significant effect on the response amplitude. This means that the dominant influence on any misfit between the actual target and the model generated by the inversion can be ascribed to factor 1 above. This situation is also evident in the profile in Fig. 6A, where the observed anomaly width is much larger than that of the calculated anomaly since the conductor is as wide as 60% of the coil separation (approximately 37 m).

The relative amplitudes and widths of positive peaks located both sides of a large negative Slingram anomaly are important in the inversion process. If the model is a dipping conductive dike, amplitudes and widths of these positive peaks become larger towards the dip direction producing an asymmetric Slingram anomaly. In the Wesley’s theory, this relationship between two positive peaks is evident for the conductor depths greater than 5 m. For the depths less than 5 m, however, the amplitude relationship is reversed. It is therefore concluded that the present inversion algorithm estimates the model parameters more correctly when depth of the target is greater than 5 m.

Although the convergence of the algorithm is not strongly affected by the selection of initial estimates of the parameters, if necessary, reasonable initial estimates could be readily made from the distance of absolute maximum value and the relative positive amplitudes of both flanks of the Slingram anomalies for \( X_0 \) and \( z \), respectively. On the other hand, the use of the anomaly amplitude may also provide a good approach for an initial depth estimation especially for the data from dual frequency surveying.

The utilization of numerical methods in the computation of partial derivatives considerably facilitates the constructing of the computer codes, although it increases the number of iterations. However, this inconvenience becomes less important as the abilities and speeds of the new generation computers are advanced. The application of numerical differentiation in iterative inversion algorithms can also facilitate to construct
codes regarding more complex conductive models such as a thin plate of finite conductivity.

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