
http://www.cecm.sfu.ca/personal/jborwein/pi_cover.html

discovered ??? convergent iteration, later found even higher order iterations. Their 1984 iteration is

\[
\alpha_0 = \sqrt{2}, \quad \beta_0 = 0, \quad \pi_0 = 2 + \sqrt{2}
\]

\[
\alpha_{n+1} = \frac{1}{2}(\alpha_n^{1/2} + \alpha_n^{-1/2}), \quad \beta_{n+1} = \alpha_n^{1/2} \left( \frac{\beta_n + 1}{\beta_n + \alpha_n} \right)
\]

\[
\pi_{n+1} = \pi_n \beta_{n+1} \left( \frac{1 + \alpha_{n+1}}{1 + \beta_{n+1}} \right) \quad \text{for which} \quad |\pi_n - \pi| \leq 10^{-2^n}.
\]

Determine the order of convergence of this iteration. What is the value of \( n \) to have at least 10 decimal digits correct for \( \pi \)?

2. Explain the formulas of a convergent Newton’s method and a fixed point iteration from geometric point of view.
3. Prove that the eigenvalues $\lambda_i, i = 1, 2, \ldots, n$ of $n \times n$ matrix $A$ lie within the union of $n$ disks $D_i$

$$|z - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n$$

in the complex plane, with center $a_{ii}$ and radius $\lambda_i$ given by the above summation.

4. Prove that rotation matrix is orthogonal.
5. Let \( v = (1, 2, 2)^T \) be the first column of a matrix. Apply one step of Householder transformation \( Q \) to find \( Qv \).

6. Prove the following theorem.

Suppose \( b \in \mathbb{R}^n \) and \( A = M - N \in \mathbb{R}^{n \times n} \) is nonsingular matrix. If \( M \) is nonsingular and the spectral radius of \( M^{-1}N \) satisfies the inequality \( \rho(M^{-1}N) < 1 \), then the iterates \( x^{(k)} \) defined by

\[
Mx^{(k+1)} = Nx^{(k)} + b
\]

converge to \( x = A^{-1}b \) for any starting vector \( x^{(0)} \).
7. Find the eigenvalues of the following matrix by Krylov’s method,

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 0 & 1 & 1 \\
0 & -1 & -2 & -2 \\
0 & 0 & 2 & 2 \\
\end{pmatrix}
\]