İzmir University of Economics, Department of Mathematics

MATH 205 ANALYTIC GEOMETRY **Solutions for Mid-Term Exam** 29.12.2003

Name **Student No.**

1. For each item, write down the corresponding conic equation from the given values

   (a) A point (x, y) on the *parabola* with focus $(h, k + p)$ and directrix $y = k - p$,

   (b) A point (x, y) on the *ellipse* with center (h, k), vertices $(h, k \pm a)$ and covertices $(h \pm b, k)$ where $c^2 = a^2 - b^2$,

   (c) A point (x, y) on the *hyperbola* with center (h, k), vertices $(h, k \pm a)$, and foci $(h, k \pm c)$ where $c^2 = a^2 + b^2$.

SOLUTION (**a**) When $\overline{x} = x - h$ and $\overline{y} = y - k$ are replaced in all of the three items above, the expressions will be more simple. In $\overline{xy}$ coordinates, the focus is $(0, 0)$ and the directrix is $\overline{y} = -p$. Thus the equation reduces to $\overline{x}^2 = 4p\overline{y}$. Now the equation of the parabola in xy coordinates becomes $(x - h)^2 = 4p(y - k)$.
(**b**) Notice that vertices are located on a vertical axis. Analogous to the above argument one may write the equation of the ellipse in $\overline{xy}$ coordinates as

$$\frac{\overline{y}^2}{a^2} + \frac{\overline{x}^2}{b^2} = 1.$$

Transforming this equation into xy coordinates gives

$$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1.$$

(**c**) Once again the vertices are located on a vertical axis.

$$\frac{\overline{y}^2}{a^2} - \frac{\overline{x}^2}{b^2} = 1.$$

Transforming this equation into xy coordinates gives

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

2. For each item in the above question, write down the related *foci, eccentricity* and *directrices* wherever applicable.

SOLUTION (**a**) The only missing information for the parabola above is the eccentricity. Recall that $|PF| = e|PD|$ and $e = 1$.
(**b**) $e = c/a$ the directrices are $y = k \pm a/e$ and the foci are $(h, k \pm c)$ since the major axis of this ellipse is vertical.
(**c**) $e = c/a$ the directrices are $y = k \pm a/e$ and the foci are $(h, k \pm c)$ since the focal axis of this hyperbola is vertical. The asymptotes are

$$y - k = \mp\frac{a}{b}(x - h).$$

3. Show that the circle centered at (h,k) with radius r is *invariant* (unchanged) under rotation transformation. (Hint: How about using translation and rotation equations?)

SOLUTION The formula for the circle is $(x-h)^2+(y-k)^2=r^2$. It is easy to work with a translated form of the circle. So, put $\bar{x}=x-h$ and $\bar{y}=y-k$ so that $\bar{x}^2+\bar{y}^2=r^2$. Now use the rotation equations

$$\begin{aligned}\bar{x} &= \hat{x}\cos\alpha-\hat{y}\sin\alpha\\ \bar{y} &= \hat{x}\sin\alpha-\hat{y}\cos\alpha\end{aligned}$$

Then

$$\bar{x}^2+\bar{y}^2=\{\hat{x}\cos\alpha-\hat{y}\sin\alpha\}^2+\{\hat{x}\sin\alpha-\hat{y}\cos\alpha\}^2=r^2.$$

Expanding and then collecting like terms on the right gives

$$\hat{x}^2\left\{\sin^2\alpha+\cos^2\alpha\right\}+\hat{y}^2\left\{\sin^2\alpha+\cos^2\alpha\right\}=r^2.$$

Since $\sin^2\alpha+\cos^2\alpha=1$ we obtain $\hat{x}^2+\hat{y}^2=r^2$ which represents a circle with radius r and centered at origin in the $\hat{x}\hat{y}$ coordinates which is obtained by rotating α angle from $\bar{x}\bar{y}$.

4. Find the length of the *astroid* curve

$$x=\cos^3 t,\qquad y=\sin^3 t\qquad 0\leqslant t\leqslant 2\pi.$$

SOLUTION Recall that the functional form of the *astroid* curve is $x^{2/3}+y^{2/3}=1$. From this, we see that the curve is symmetric with respect to x axis, y axis and the origin. So we just consider the part of the curve where $t\in[0,\pi/2]$. Then the whole length L is 4 times of the specific part.

$$L=4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

We have

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2&=(3\cos^2 t(-\sin t))^2=9\cos^4 t\sin^2 t\\ \left(\frac{dy}{dt}\right)^2&=(3\sin^2 t(\cos t))^2=9\sin^4 t\cos^2 t\end{aligned}$$

and

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9\sin^2 t\cos^2 t(\sin^2 t+\cos^2 t)}=3|\sin t\cos t|.$$

Therefore the length is

$$L=4\int_0^{\pi/2}3|\sin t\cos t|dt=4\int_0^{\pi/2}\frac{3}{2}\sin 2t dt,$$

since both $\sin t$ and $\cos t$ are nonnegative for $t\in[0,\pi/2]$. Thus

$$L=-3\left[\cos 2t\right]_{t=0}^{t=\pi/2}=(-3)(-2)=6.$$

5. Draw the graph of the conic

$$5x^2 + 4xy + 2y^2 - 24x - 12y + 18 = 0,$$

and find *center, foci,vertices, eccentricity, directrix* of the new graph.

SOLUTION The coefficients of the cone are

$$A = 5, \quad B = 4, \quad C = 2, \quad D = -24, \quad E = -12, \quad F = 18.$$

First, let us find the angle for rotation.

$$\cot 2\alpha = \frac{5-2}{4} \quad \text{and} \quad \sin\alpha = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\alpha = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}.$$

Using the formulas we find the coefficients (or replace the rotation equation in the ellipse)

$$A' = \frac{4}{5}A + \frac{2}{5}B + \frac{1}{5}C = 6, \quad B' = 0, \quad C' = \frac{1}{5}A - \frac{2}{5}B + \frac{4}{5}C = 1$$
$$D' = \frac{2}{\sqrt{5}}D + \frac{1}{\sqrt{5}}E = -12\sqrt{5}, \quad E' = -\frac{1}{\sqrt{5}}D + \frac{2}{\sqrt{5}}E = 0, \quad F' = F.$$

The curve is now

$$6x'^2 + y'^2 - 12\sqrt{5}x' + 18 = 0 \Longleftrightarrow 6(x' - \sqrt{5})^2 + y'^2 = 12 \Longleftrightarrow \frac{(x' - \sqrt{5})^2}{2} + \frac{y'^2}{12} = 1.$$

From the last expression we see that the curve is an *ellipse* whose focal axis is vertical in the new coordinates $x'y'$ since the coefficient of y',$a^2 = 12 > b^2 = 2$. The *center* is $(\sqrt{5}, 0)$ and the focal distance $c = \sqrt{12-2} = \sqrt{10}$. So the foci are $(\sqrt{5}, \pm\sqrt{10})$ and *vertices* $(\sqrt{5}, \pm 2\sqrt{3}$. The *eccentricity* e is $e = c/a = \sqrt{10}/2\sqrt{3}$. The *directrices* are

$$y' = \pm\frac{a}{e} = \frac{a^2}{c} = \frac{12}{\sqrt{10}}.$$

Note that to find the related information in the xy coordinates using the values in the $x'y'$ coordinates one should use the rotation equations with the angle α replaced by $-\alpha$.