İzmir University of Economics, Department of Mathematics

MATH 205 ANALYTIC GEOMETRY **Exam 1** 12.12.2004

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You will not get any points if your answer is wrong, that is no points to your explanations if your answer is wrong. And of course no points to a correct answer if your explanation or proof is not correct or clear.

YOU must write GOOD Mathematics

1. Find the parametric equation of *circle*, *ellipse* and *hyperbola* centered at (x_0, y_0) using their standard equations.
SOLUTION:

Circle: $(x-x_0)^2+(y-y_0)^2=r^2$ (3), set $x-x_0=r\cos t$, (3) $y-y_0=r\sin t$, $0\leq t\leq 2\pi$

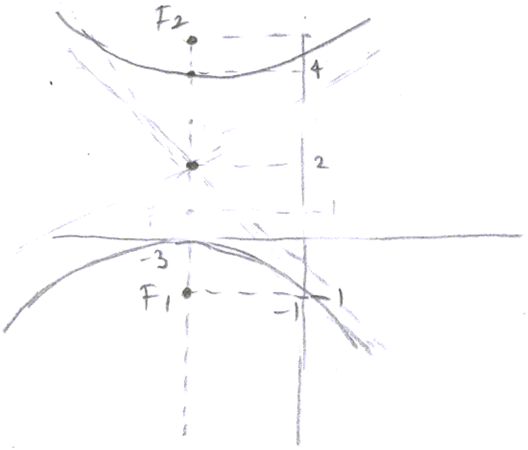
since $r^2\cos^2 t+r^2\sin^2 t=r^2$

Ellipse: $\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}=1$ (3). Set $x-x_0=a\cos t$, $y-y_0=b\sin t$

(4) since $\dfrac{a^2\cos^2 t}{a^2}+\dfrac{b^2\sin^2 t}{b^2}=1$, $0\leq t\leq 2\pi$.

Hyperbola $\dfrac{(x-x_0)^2}{a^2}-\dfrac{(y-y_0)^2}{b^2}=1$, (3) Set $x-x_0=\mp a\sec t$, $y-y_0=a\tan t$ (4) $-\frac{\pi}{2}<t<\frac{\pi}{2}$

2. Write the equation of the hyperbola whose center is $(-3,2)$, one vertex is $(-3,4)$ and one focus is $(-3,-1)$. Then find the *eccentricity* and *directrices* and *asymptotes*.
SOLUTION:



The other vertex is $(-3,0)$, by equidistance.

(3) So, $a=2$, center-vertex distance

(3) $c=3$, center-focus distance. Thus $c=\sqrt{a^2+b^2}$, $b^2=5$.

$x_0=-3$, $y_0=2$ and the equation becomes

(5) $\dfrac{(y-2)^2}{4}-\dfrac{(x+3)^2}{5}=1$, $e=\dfrac{c}{a}=\dfrac{3}{2}$ (3)

directrices $y-2=\mp\dfrac{3}{4}$ (3)

asymtotes $\dfrac{(y-2)^2}{4}=\dfrac{(x+3)^2}{5}$ or $y-2=\mp\dfrac{2}{\sqrt{5}}(x+3)$ (3)

3. Use rotation of axes to find the standard equation of the conic $x^2 + Bxy + y^2 = 1$. Then find the *eccentricity*

SOLUTION:

$$\cot 2\theta = \frac{A-C}{2B} = \frac{1-1}{2B} = 0, \quad 2 \qquad 2\theta = 90^\circ, \quad \theta = 45^\circ \text{ or } \frac{\pi}{4}$$

Set

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad \text{or} \quad 6 \quad \begin{aligned} x &= \frac{1}{\sqrt{2}}(\tilde{x} - \tilde{y}) \\ y &= \frac{1}{\sqrt{2}}(\tilde{x} + \tilde{y}) \end{aligned}$$

Then

$$x^2 + Bxy + y^2 = \frac{1}{2}(\tilde{x} - \tilde{y})^2 + B \cdot \frac{1}{\sqrt{2}}(\tilde{x} - \tilde{y}) \frac{1}{\sqrt{2}}(\tilde{x} + \tilde{y}) + \frac{1}{2}(\tilde{x} + \tilde{y})^2 = 1$$

or

$$= \tilde{x}^2\left(1 + \frac{B}{2}\right) + \tilde{y}^2\left(1 - \frac{B}{2}\right) = 1 \quad 8$$

(i) if it is ellipse; i.e. $B \in (-2, 2)$ then

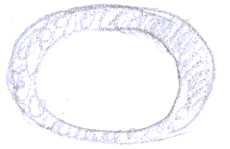
$$a^2 = \frac{1}{1 + \frac{B}{2}} \quad \text{and} \quad b^2 = \frac{1}{1 - B/2} \quad \text{and} \quad c^2 = \frac{1}{1 + B/2} - \frac{1}{1 - B/2} = \frac{\left(1 - \frac{B}{2}\right) - \left(1 + \frac{B}{2}\right)}{1 - \frac{B^2}{4}} = \frac{4B}{B^2 - 4}$$

and $e = \sqrt{\frac{4B}{B^2-4} \cdot \left(1 + \frac{B}{2}\right)} = \sqrt{\boxed{\frac{2B}{B-2}}}$ with $B > 0$.

4. Find the surface area A of the torus generated by revolving the circle $(x-a)^2 + y^2 = r^2$ about the y axis.

SOLUTION:

(ii) Otherwise it is a line //, degenerate conic.



The area A has formula

$$A = 2\pi \int_0^{2\pi} x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad 4$$

of a function $x = x(t)$, $y = y(t)$, $0 < t < 2\pi$ about y-axis.

The circle has parametrization of the circle is

$$x = a + r\cos t, \quad y = r\sin t \quad 0 < t < 2\pi. \quad 4$$

So $\frac{dx}{dt} = -r\sin t$, $\frac{dy}{dt} = r\cos t$

Thus, substituting these values in the above equation yields

$$A = 2\pi \int_0^{2\pi} (a + r\cos t) \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \quad 12$$

$$= 2\pi \int_0^{2\pi} (a + r\cos t)\, r\, dt \quad \text{since } \sin^2 t + \cos^2 t = 1.$$

Then

$$A = 2\pi \left\{ ar \int_0^{2\pi} dt + r^2 \int_0^{2\pi} \cos t\, dt \right\}$$

$$= 4\pi^2 ar + r^2 \left[\sin t\right]_{t=0}^{t=2\pi} = 4\pi^2 ar$$

5. Verify that one complete oscillation of the sine curve $y = \sin x$ has the same length L as the ellipse $\frac{1}{2}x^2 + y^2 = 1$.

SOLUTION:

The curve $y=f(x)=\sin x$ has length L, $L = \int_0^{2\pi} \sqrt{1+(f'(x))^2}\, dx$ 8

since one oscillation is completed in an interval of length 2π for $\sin x$ (or $\cos x$). Then we find $L = \int_0^{2\pi} \sqrt{1+\cos^2 x}\, dx$ //.

The ellipse has parametrization

$x = \sqrt{2}\cos t$, $y = \sin t$, $0 \le t \le 2\pi$ so that $\frac{x^2}{2} + y^2 = 1$.

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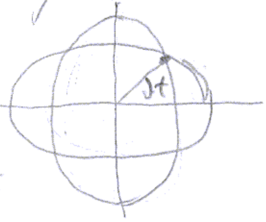
This ellipse has the same length with

$x = \sqrt{2}\sin t$, $y = \cos t$, $0 \le t \le 2\pi$



$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

with $x'(t) = \sqrt{2}\cos t$, $y'(t) = -\sin t$

$= \int_0^{2\pi} \sqrt{2\cos^2 t + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{\cos^2 t + 1}\, dt$ //

6. (a) Show that the "*Eight Curve*" $x^4 = a^2(x^2 - y^2)$ has polar equation $r^2 = a^2\cos(2\theta)\sec^4\theta$.

(b) Find the symmetries of this curve.

SOLUTION:

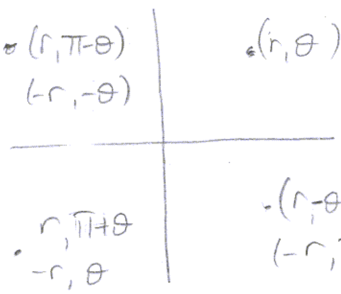
(a) Let $x = r\cos\theta$; $y = r\sin\theta$ and $x^2+y^2=r^2$

Then $r^4\cos^4\theta = a^2(r^2\cos^2 - r^2\sin^2\theta)$, cancelling r^2

$r^2\cos^4 = a^2\cos 2\theta$ or 10

$r^2 = a^2\cos 2\theta / \cos^4\theta$, or $r^2 = a^2\cos 2\theta\sec^4\theta$

(b) $F(r,\theta) = r^2 - a^2\cos 2\theta\sec^4\theta = 0$ 10



F is symmetric w.r.t. origin since

$F(-r,\theta) = (-r)^2 - a^2\cos 2\theta\sec^4\theta = F(r,\theta)$

F is symmetric w.r.t. x-axis since

$F(r,-\theta) = r^2 - a^2\cos(-2\theta)\sec^4(-\theta) = F(r,-\theta)$

F is symmetric w.r.t y-axis since

$F(-r,-\theta) = (-r)^2 - a^2\cos(-2\theta)\sec^4(-\theta) = F(r,\theta)$