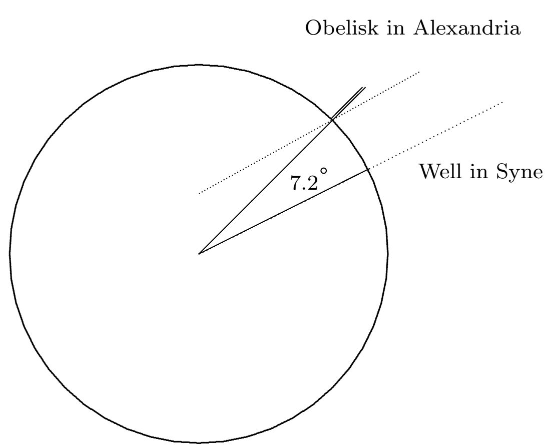
The size of the Earth

Eratosthenes (275-195 B.C.) head of the famous library at Alexandria in Egypt made the first measurement of the Earth's size. His demonstration is one of the most beautiful ever performed. Because it so superbly illustrates *how science links observation and logic*.

Astronomers were very acquainted with the yearly movement of the Sun and could predict accurately the times of the solstices and equinoxes. Eratosthenes, a geographer as well as astronomer, heard that lying to the south, Syene (the present city of Aswan), the Sun would be directly overhead at noon and cast no shadow. Sun shone exactly down a well. Knowing the distance between Alexandria and Syene and the power of geometry, he realized he could deduce the circumference of the Earth. He analyzed the problem as follows:

- *Sun's rays are parallel lines* Because the Sun is far away from the Earth, its light travels in parallel rays toward the Earth.
- *Parallel lines form equal angles* Imagine drawing a straight line from the center of the Earth outward so that it passes vertically through the Earth's surface in Alexandria. The angle between that line and the Sun's rays in Alexandria is the same as the angle between that line and the line from the center of the Earth up through the well in Syene.
- The cities Alexandria and Syene are in the same longitude.

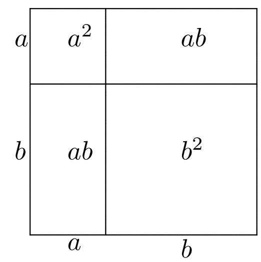
He measured that angle between the direction to the Sun and the vertical to the ground with sticks and a protractor to be 7.2°.



Since he knew the distance between Alexandria and the well in Syene, to be 5000 stadia (measure or norm used to be stadium), now 800kms, the circumference L of the Earth is 50 times this distance. That is 40000kms. (actual value 40074kms)

$$\frac{7.2}{360} = \frac{800}{L}.$$

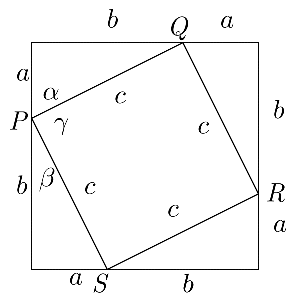
Pythagorean Theorem *Pythagoras of Samos* (380-300 B.C.)
Algebraic identities were verified by geometric figures. For example, $(a+b)^2 = a^2 + 2ab + b^2$.



In a right triangle with legs of length a and b and hypothenuse of length c, we have

$$c^2 = a^2 + b^2.$$

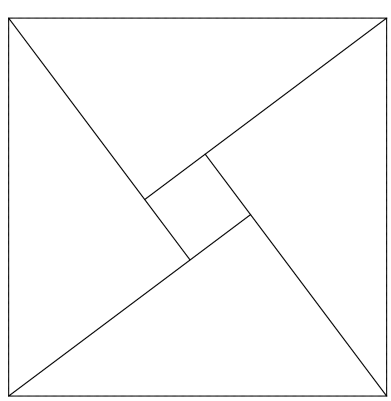
The converse is also true, it must be a right triangle.



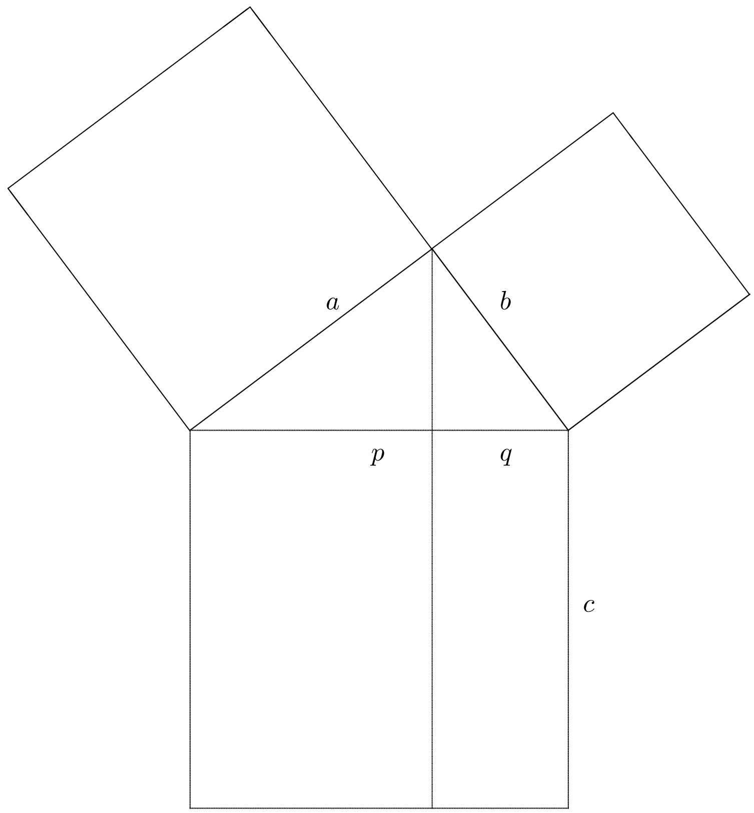
The figure is a square with side lengths $a+b$. Inside the square is a rhombus $PQRS$ with side length c. We claim the rhombus is a square. Notice that the triangles inside the square are congruent. The interior angle at P is $\alpha + \gamma + \beta = 180°$ since $\alpha + \beta = 90°$, acute angles in a right triangle, so $\gamma = 90°$. Thus, the rhombus is a square. Now, we write the area of the outer square as a sum of 4 congruent triangles and inner square,

$$(a+b)^2 = c^2 + 4\frac{1}{2}ab \quad \Longleftrightarrow \quad a^2 + b^2 = c^2.$$

Next we give another proof for the theorem. In the figure, a square of side length c is partitioned with a square of side length $a-b$ in the center. Complete the proof.



The Pythagorean theorem given in the Euclid's (about 325-265 B.C.) "Elements" is based on the following figure.



The big square is divided into two rectangles whose areas are pc and qc. We must show that $a^2 = pc$ and $b^2 = qc$. Corresponding sides of these similar triangles are the proportions

$$\frac{p}{a} = \frac{a}{c} \quad \text{and} \quad \frac{q}{b} = \frac{b}{c}.$$

Hence, we have

$$a^2 + b^2 = pc + qc = c(p + q) = c^2.$$

The book by Elista Scott Loomis "Pythagorean Proposition" contains 370 proofs of the theorem.

Euclid's most famous work is his treatise on mathematics *The Elements*. The book was a compilation of knowledge that became the centre of mathematical teaching for 2000 years. Probably no results in The Elements were first proved by Euclid but the organisation of the material and its exposition are certainly due to him. In fact there is ample evidence that Euclid is using earlier textbooks as he writes the Elements since he introduces quite a number of definitions which are never used such as that of an oblong, a rhombus, and a rhomboid.

The Elements begins with definitions and five postulates. The first three postulates are postulates of construction, for example the first postulate states that it is possible to draw a straight line between any two points. These postulates also implicitly assume the existence of points, lines and circles and then the existence of other geometric objects are deduced from the fact that these exist. There are other assumptions in the postulates which are not explicit. For example it is assumed that there is a unique line joining any two points. Similarly postulates two and three, on producing straight lines and drawing circles, respectively, assume the uniqueness of the objects the possibility of whose construction is being postulated.

The fourth and fifth postulates are of a different nature. Postulate four states that all right angles are equal. This may seem "obvious" but it actually assumes that space in homogeneous - by this we mean that a figure will be independent of the position in space in which it is placed. The famous fifth, or parallel, postulate states that one and only one line can be drawn through a point parallel to a given line. Euclid's decision to make this a postulate led to Euclidean geometry. It was not until the 19th century that this postulate was dropped and non-euclidean geometries were studied.

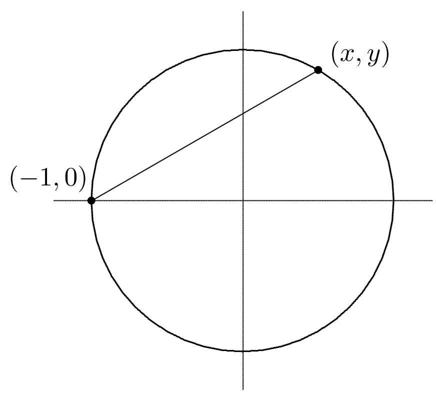
We describe all the integer solutions to $a^2+b^2=c^2$; The Pythagorean triples (a,b,c). Almost everybody is familiar with $3^2+4^2=5^2$. The history of the Pythagorean triples goes back to Babylonians about 1800 B.C. Chine about 200 B.C. and India about 500 B.C. It is interesting to note that Babylonians knew the triple (12709,13500,18541).
Theorem *Pythagorean triples* are the integer multiples of triples of the from

$$(2rs, r^2-s^2, r^2+s^2), \qquad \text{or} \qquad (r^2-s^2, 2rs, r^2+s^2)$$

where r,s are integers.
Proof Since $(2rs)^2+(r^2-s^2)^2=(r^2+s^2)^2$ are indeed Pythagorean triples. Notice that all integer multiples of the triples are also in the set since if we multiply each element of such a triple by n, then we multiply the equality by n^2 on both sides. We must prove that this characterizes every Pythagorean triple. Suppose a,b,c are integers with $a^2+b^2=c^2$ and $b,c\neq 0$. We can normalize this by dividing by c^2 and hence

$$x^2+y^2=1, \qquad \text{where} \qquad x=\frac{a}{c}, y=\frac{b}{c}.$$

The graph of the latter equation is the unit circle, that is the circle of radius 1 with center at the origin. Let us call a point (x,y) in the plane a rational point if both its coordinates are rational numbers. Pythagorean triples are essentially the same as that of finding all rational points (x,y) on the unit circle. (First quadrant if both x,y are positive) It is helpful to describe the circle using parametric equations, where the parameter t is the slope of the line from $(-1,0)$ to (x,y).



The slope of the line is $t=y/(1+x)$ and is a rational number because both x,y are rational numbers. Substituting $y=t(1+x)$ into $x^2+y^2=1$ gives $x^2+t^2(1+x)^2=1$ which we rewrite as

$$x^2+\frac{2t^2}{1+t^2}x+\frac{t^2-1}{1+t^2}=0.$$

We observe that the first coordinate x, of a point on both the line and the circle must satisfy the above quadratic equation. If the roots are, say α and β, then we have

$$x^2+\frac{2t^2}{1+t^2}x+\frac{t^2-1}{1+t^2}=(x-\alpha)(x-\beta).$$

The product of the roots is

$$\alpha\beta=\frac{t^2-1}{1+t^2}.$$

From the point $(-1,0)$, since $x=-1$ is one of the roots the other root must be $(1-t^2)/(1+t^2)$. Again, notice that this root must be rational, since t is rational. Let $t=s/r$ in lowest terms. We obtain the solution with respect to s,r,

$$x=\frac{1-t^2}{1+t^2}=\frac{1-s^2/t^2}{1+s^2/t^2}=\frac{r^2-s^2}{r^2+s^2} \qquad \text{and} \qquad y=t(1+x)=\frac{s}{r}\frac{2r^2}{r^2+s^2}=\frac{2rs}{r^2+s^2}.$$

This gives us the Pythagorean triple

$$(r^2 - s^2, 2rs, r^2 + s^2)$$

where the ratios equal the original ratios a/c and b/c. Because the ratios are the same, the original triple (a, b, c) is a multiple of this triple by a rational number q. If q is an integer, then we have the desired conclusion. Otherwise, the triple has a common factor $n \neq \pm 1$. Hence n also divides the sum and difference of $r^2 - s^2$ and $r^2 + s^2$. Those are $2r^2$ and $2s^2$ respectively. Since r and s have no common factors, this requires $n = \pm 2$. Hence the terms of the triple are all even and r, s must both be odd. Now let $R = (r + s)/2$ and $S = (r - s)/2$. We compute

$$2RS = \frac{r^2 - s^2}{2}, \quad R^2 - S^2 = rs, \quad R^2 + S^2 = \frac{r^2 + s^2}{2}$$

and hence $(a, b, c) = (2RS, R^2 - S^2, R^2 + S^2)$ which is again the desired form.

This is the elegant strategy of *Diophantus* (about 200 B.C. in Alexandria) .

In the 1630's P. Fermat (1601,1675) wrote in the margin of his copy of a book by Diophantus,

> . . . it is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as a sum of two fourth powers or, in general, for any number which is a power greater than the second to be written as a sum of two like powers. I have discovered a truly marvellous demonstration of this proposition which this margin is too narrow to contain.

Fermat never wrote a proof except for $n = 4$. The statement "For every $n > 2$, there are no positive integers a, b, c with $a^n + b^n = c^n$ " is known as *Fermat's Last Theorem.* Many important techniques were developed in unsuccessful attempts to prove it. At last, in 1995, *Andrew Wiles* with the help of Richard Taylor succeeded in finding a proof that had eluded mathematicians over 350 years. His paper appeared in "Modular Elliptic Curves and Fermat's Last Theorem", *Annals of Mathematics*, **141**, (1995), 443-551.

References

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