
MATH 223 LINEAR ALGEBRA

Final Exam

04.01.2005

Name

Student No.

Sign

You will not get any points if your answer is wrong, that is no points to your explanations if your answer is wrong. And of course no points to a correct answer if your explanation or proof is not correct or clear.

YOU must write GOOD Mathematics

IN QUESTIONS NUMBER 2,3, AND 4, SOLVE ONLY ANY 2 OF THEM.

1. Let $V = \mathbb{P}_3(\mathbb{R})$ (real polynomials of degree 3) and $T(f(x)) = f(x) + f(2)x$. Find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

SOLUTION:

IN THIS PAGE, SOLVE ONLY ANY 2 OF THE QUESTIONS.

2. Let λ be an eigenvalue of a linear operator T and let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a nonzero polynomial. Show that $f(\lambda)$ is an eigenvalue of the operator $f(T)$.

SOLUTION:

3. Prove that every eigenvalue of a *self-adjoint* operator T on a finite dimensional inner product space V is *real*.

SOLUTION:

4. Let V be an inner product space and $S = \{v_1, v_2, \dots, v_n\}$ be an *orthogonal* subset of V consisting of nonzero vectors. Prove that if $f \in \text{span}(S)$, then

$$f = \sum_{i=1}^n \frac{\langle f, v_i \rangle}{\|v_i\|^2} v_i.$$

5. Let V be a vector space with $\dim(V) = 8$ and let T be a linear operator whose minimal polynomial $m(t)$ is $m(t) = (t^2 + t + 1)^2(t + 1)^2$. Represent all possible (without permuting) rational canonical form of T .

SOLUTION:

6. Consider the hyperbola $xy = 1$;

- (a) Write the hyperbola in the form $z^T A z$ where $z \in \mathbb{R}^2$ is a column vector and $A \in \mathbb{R}^{2 \times 2}$.
- (b) Find an *orthogonal* matrix Q which rotates xy coordinate system to $\hat{x}\hat{y}$ system and a *diagonal* matrix D such that $A = QDQ^T$.
- (c) Find the equation of the hyperbola in the new coordinate system $\hat{x}\hat{y}$

SOLUTION:

7. For the following matrix A whose characteristic polynomial $f(t)$ is $f(t) = -(t - 1)(t - 2)^2$, find a *Jordan canonical form* J and an invertible matrix Q such that $J = Q^{-1}AQ$,

$$A = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{pmatrix}$$

SOLUTION:

Isaac Newton born today (1643 - 1727), "If I have been able to see further, it was only because I stood on the shoulders of giants."