**Name-Surname:**
**Student No:** **Class:** **Grade:**

**MAT435** **Computer Aided Geometric Design**

# Answers for Exam 1

**09.12.2003**

**1.** Given $n+1$ points $\mathsf{b}_0, \mathsf{b}_1, \ldots, \mathsf{b}_n$ in the Euclidean space $\mathbb{E}^2$ or $\mathbb{E}^3$, set $\mathsf{b}_i^0 = \mathsf{b}_i, i = 0, 1, \ldots, n$, the *de Casteljau algorithm* is

$$\mathsf{b}_i^r(t) = (1-t)\mathsf{b}_i^{r-1}(t) + t\mathsf{b}_{i+1}^{r-1}(t) \quad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r. \end{cases}$$

Prove by induction on $r$ that the intermediate de Casteljau points $\mathsf{b}_i^r$ can be represented by

$$\mathsf{b}_i^r(t) = \sum_{j=0}^{r} \mathsf{b}_{i+j} B_j^r(t), \quad r \in \{0, 1, \ldots, n\},\ i \in \{0, 1, \ldots, n-r\}.$$

Solution  At $r = 0$, $\mathsf{b}_i^0(t) = \mathsf{b}_i$ since $j = 0$ and $B_0^0(t) = 1$. Suppose it is true for $r \geqslant 1$. Then from the algorithm

$$\mathsf{b}_i^{r+1}(t) = (1-t)\sum_{j=0}^{r} \mathsf{b}_{i+j} B_j^r(t) + t\sum_{j=0}^{r} \mathsf{b}_{i+j+1} B_j^r(t).$$

For $1 \leqslant j \leqslant r$, the coefficient of the point $\mathsf{b}_{i+j}$ in the two summations above are

$$(1-t)B_j^r(t) + tB_{j-1}^r(t) = B_j^{r+1}(t).$$

The initial point $\mathsf{b}_i$ comes from the first summation and the end point $\mathsf{b}_{i+r+1}$ comes from the second summation. Thus we obtain

$$\mathsf{b}_i^{r+1}(t) = \sum_{j=0}^{r} \mathsf{b}_{i+j} B_j^{r+1}(t)$$

which completes the proof.

**2.** Find the transformation matrix $\mathbf{T}$ between the monomial basis $\Psi = \{1, t, \ldots, t^n\}$ and Bernstein-Bézier basis $\Phi = \{B_0^n(t), B_1^n(t), \ldots, B_n^n(t)\}$ of $\mathcal{P}_n$ such that $\Psi = \mathbf{T}\Phi$, where

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \ldots, n, \quad t \in [0, 1].$$

(Finding only for $n = 3$ gets 10 percent of the grade)

Solution  Since $1 = (t + (1-t))^{n-i} = \sum_{j=0}^{n-i} B_j^{n-i}(t)$, we have

$$t^i = \sum_{j=0}^{n-i} \binom{n-i}{j} t^{i+j}(1-t)^{n-i-j} = \sum_{j=i}^{n} \binom{n-i}{j-i} t^j (1-t)^{n-j},$$

by shifting the index of the summation. We may write the binomial coefficient

$$\binom{n-i}{j-i} = \frac{(n-i)!}{(j-i)!(n-j)!} = \frac{n!}{n!}\frac{i!}{i!}\frac{j!}{j!}\frac{(n-i)!}{(j-i)!(n-j)!} = \frac{\binom{j}{i}}{\binom{n}{i}}\binom{n}{j}.$$

Thus we deduce that

$$t^i = \sum_{j=i}^{n} \frac{\binom{j}{i}}{\binom{n}{i}} B_j^n(t), \quad \text{for} \quad i = 0, 1, \ldots, n$$

and the transformation matrix $\mathbf{T}$ has elements $T_{ij}$ satisfying

$$\mathbf{T}_{ij} = \frac{\binom{j}{i}}{\binom{n}{i}}, i, j = 0, 1, \ldots, n$$

and is upper triangular. (This matrix is the inverse of the matrix $\mathbf{M}$ that we have done in one of the lectures)

**3.** Let $\mathsf{b}_0 = (-1,0), \mathsf{b}_1 = (-3,3), \mathsf{b}_2 = (3,3), \mathsf{b}_3 = (1,0)$ be the control points and $\mathsf{P}(t)$ be Bézier curve. Degree elevate the points and hence find a new set of control points without disturbing the curve. Draw both of the control polygons.

SOLUTION On comparing the coefficient the basis in the equation

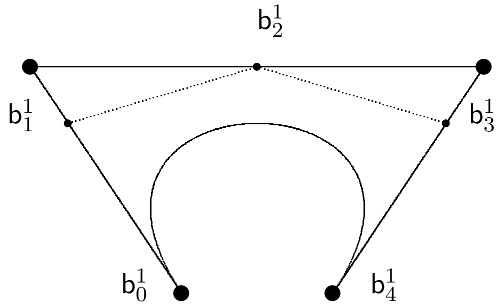
$$\mathsf{P}(t) = \sum_{i=0}^{n=3} \mathsf{b}_i \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n+1} \mathsf{b}_i^1 \binom{n+1}{i} t^i (1-t)^{n+1-i}$$

We obtain the degree elevated control points

$$\mathsf{b}_i^1 = \frac{i}{n+1}\mathsf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathsf{b}_i, \quad i = 0, 1, \ldots, n+1.$$

Thus it is calculated from the latter equation that

$$\mathsf{b}_0^1 = \mathsf{b}_0, \quad \mathsf{b}_1^1 = (-10/4, 9/4), \quad \mathsf{b}_2^1 = (0,3), \quad \mathsf{b}_3^1 = (10/4, 9/4), \mathsf{b}_4^1 = \mathsf{b}_3.$$




*A beautiful picture of cutting corners*

**4.** Find explicitly the blossom representation of a cubic Bézier curve.

SOLUTION Given the control points, $\mathsf{b}_0, \mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3$, the method to find the blossom representation is to insert a new variable $t_r$ at the $r$th step of the de Casteljau algorithm. Thus it results the following triangular array.

$$\begin{array}{llll}
\mathsf{b}_0 & (1-t_1)\mathsf{b}_0 + t_1\mathsf{b}_1 := \mathsf{b}_0^1[t_1] & (1-t_2)\mathsf{b}_0^1[t_1] + t_2\mathsf{b}_1^1[t_1] := \mathsf{b}_0^2[t_1,t_2] & (1-t_3)\mathsf{b}_0^2[t_1,t_2] + t_3\mathsf{b}_1^2[t_1,t_2] := \mathsf{b}_0^3[t_1,t_2,t_3] \\
\mathsf{b}_1 & (1-t_1)\mathsf{b}_1 + t_1\mathsf{b}_2 := \mathsf{b}_1^1[t_1] & (1-t_2)\mathsf{b}_1^1[t_1] + t_2\mathsf{b}_2^1[t_1] := \mathsf{b}_1^2[t_1,t_2] & \\
\mathsf{b}_2 & (1-t_1)\mathsf{b}_2 + t_1\mathsf{b}_3 := \mathsf{b}_2^1[t_1] & & \\
\mathsf{b}_3 & & &
\end{array}$$

This is a symmetric multi-affine mapping $(t_1, t_2, t_3) \longmapsto t : \mathbb{R}^3 \longmapsto \mathbb{R}$. Explicitly

$$\begin{aligned}
\mathsf{b}_0^3[t_1,t_2,t_3] = (1-t_1)(1-t_2)(1-t_3)\mathsf{b}_0 + \{t_1(1-t_2)(1-t_3) + t_2(1-t_1)(1-t_3) + t_3(1-t_1)(1-t_2)\}\,\mathsf{b}_1 \\
+ \{t_1t_2(1-t_3) + t_2t_3(1-t_1) + t_1t_3(1-t_2)\}\,\mathsf{b}_2 + t_1t_2t_3\mathsf{b}_3
\end{aligned}$$

Now, the cubic Bézier is attained at the diagonal of blossom $\mathsf{b}_0^3[t,t,t]$. That is

$$\mathsf{P}(t) = \mathsf{b}_0^3[t,t,t] = \sum_{j=0}^{n=3} \mathsf{b}_j B_j^n(t)$$

Even the control points are special points of the blossom

$$\mathsf{b}_0 = \mathsf{b}_0^3[0,0,0] \qquad \mathsf{b}_1 = \mathsf{b}_0^3[0,0,1]$$
$$\mathsf{b}_2 = \mathsf{b}_0^3[0,1,1] \qquad \mathsf{b}_3 = \mathsf{b}_0^3[1,1,1]$$

*As beautiful as a blossom of a rose*