**Name-Surname:**
**Student No:** **Class:** **Grade:**

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**MAT435** **Computer Aided Geometric Design**

# Answers for Quiz 3 **18.11.2003**

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**YOU must write GOOD Mathematics, clearly explaining each step of your proof. Otherwise, no objection!**

**1.** Prove by induction on $k$ that

$$\Delta^k \mathbf{b}_i = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathbf{b}_{i+k-j}.$$

Solution We prove it by induction $k$. The initial case $k = 0$ is clear. Let us suppose that it is true for any integer $k \geqslant 1$ and try to obtain the order $k+1$th from that of $k$th. For this purpose let us remark that

$$\Delta^{k+1}\mathbf{b}_i = \Delta^k(\Delta \mathbf{b}_i) = \Delta^k(\mathbf{b}_{i+1} - \mathbf{b}_i) = \Delta^k \mathbf{b}_{i+1} - \Delta^k \mathbf{b}_i.$$

We want to deduce that the coefficient of $\mathbf{b}_{i+k+1-j}$ in the expansion of $\Delta^{k+1}\mathbf{b}_i$ is equal to $(-1)^j\binom{k+1}{j}$. Thus applying the formula to the RHS of the equation of the induction hypothesis gives

$$\Delta^{k+1}\mathbf{b}_i = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathbf{b}_{i+k+1-j} - \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathbf{b}_{i+k-j}.$$

The coefficient of the term $\mathbf{b}_{i+1+k-j}$ for $1 \leqslant j \leqslant k$ on the two summations above is

$$(-1)^j \binom{k}{j} - (-1)^{j-1} \binom{k}{j-1},$$

where $j$ is replaced by $j-1$ to get the term $\mathbf{b}_{i+1+k-j}$ from $\mathbf{b}_{i+k-j}$. These two binomial coefficients are combined using Pascal's identity to give the coefficient of the term $\mathbf{b}_{i+1+k-j}$, $(-1)^j\binom{k+1}{j}$. Here notice that for $j = 0$ only the first binomial is nonzero and for $j = k+1$ the only nonzero binomial is the second. These two conditions establish the end values of the summation in the expansion $\Delta^{k+1}\mathbf{b}_i$. Thus the induction is complete

$$\Delta^{k+1}\mathbf{b}_i = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \mathbf{b}_{i+k+1-j}.$$

**2.** Given the control points $(0,0), (2,2), (4,0), (6,2)$, let $\mathbf{X}(t)$ be the Bézier curve. Evaluate $\ddot{\mathbf{X}}(t)$ (the second derivative), *using the derivative formula for the Bézier curves.* (Direct derivative calculation will not be accepted in any circumstance)

Solution We have the $k$th derivative formula

$$\mathbf{X}^{(k)}(t) = \frac{n!}{(n-k)!} \sum_{i=0}^{n-k} \Delta^k \mathbf{b}_i B_i^{n-k}(t).$$

Let us evaluate the differences,

$$(0,0), (2,2), (4,0), (6,2)$$
$$\Delta \mathbf{b}_i; (2,2), (2,-2), (2,2)$$
$$\Delta^2 \mathbf{b}_0 = (0,-4), \Delta^2 \mathbf{b}_1 = (0,4)$$

Thus applying the formula and placing the difference we obtain

$$\ddot{\mathbf{X}}(t) = 6((1-t).(0,-4) + t.(0,4)) = (0, 48t - 24).$$