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MATH 435 COMPUTER AIDED GEOMETRIC DESIGN **Answers for Quiz 5** 30.12.2003

Name **Student No.**

YOU must write GOOD Mathematics, clearly explaining each step of your proof. Otherwise, no objection will be accepted.

1. Given three points $\mathsf{p}_0, \mathsf{p}_1$ and p_2 in $\mathbb{R}^2$. Find the Bézier points $\mathsf{b}_0, \mathsf{b}_1$ and b_2, and $\mathsf{x}(t)$ so that the Bézier curve $\mathsf{x}(t)$ passes through (*interpolate*) the data points. (*Hint: Set the parameter values* $t_i = i/n$)

SOLUTION We want to find a curve $\mathsf{x}(t)$ satisfying

$$\mathsf{x}(t) = \sum_{i=0}^{n=2} \mathsf{b}_i B_i^n(t), \qquad \mathsf{x}(t_0) = \mathsf{p}_0, \mathsf{x}(t_1) = \mathsf{p}_1, x(t_2) = \mathsf{p}_2.$$

In order to determine the Bézier points, we select $t_0 = 0, t_1 = 1/2$ and $t_2 = 1$ (uniform knot sequence makes calculations easy). Following the end point interpolation property of the basis functions, $t = 0$ gives $\mathsf{b}_0 = \mathsf{p}_0$ and $t = 1$ gives $\mathsf{b}_2 = \mathsf{p}_2$. At the parameter value $t = 1/2$, we have

$$\frac{1}{4}\mathsf{b}_0 + \frac{1}{2}\mathsf{b}_1 + \frac{1}{4}\mathsf{b}_2 = \mathsf{p}_1$$

which yields

$$\mathsf{b}_1 = \frac{1}{2}(-4\mathsf{p}_0 + 4\mathsf{p}_1 - \mathsf{p}_2).$$

Finally, the interpolating polynomial for these parameter values is

$$\mathsf{x}(t) = \mathsf{p}_0 B_0^2(t) + \frac{1}{2}(-4\mathsf{p}_0 + 4\mathsf{p}_1 - \mathsf{p}_2)B_1^2(t) + \mathsf{p}_0 B_2^2(t).$$

Note that a linear system of equation could also be used to solve the problem. Namely, solve

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathsf{b}_0 \\ \mathsf{b}_1 \\ \mathsf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{p}_0 \\ \mathsf{p}_1 \\ \mathsf{p}_2 \end{bmatrix}$$

2. Given two points $\mathsf{p}_0, \mathsf{p}_1$ and a tangent vector $\mathbf{m}_0$ at p_0 in $\mathbb{R}^2$. Find the Bézier points $\mathsf{b}_0, \mathsf{b}_1$ and b_2, and $\mathsf{x}(t)$ so that the Bézier curve $\mathsf{x}(t)$ passes through (*interpolate*) the data points.

SOLUTION We want to find a curve, (namely the Hermite interpolating curve) $\mathsf{x}(t)$ satisfying

$$\mathsf{x}(t) = \sum_{i=0}^{n=2} \mathsf{b}_i B_i^n(t), \qquad \mathsf{x}(0) = \mathsf{p}_0, \mathsf{x}(1) = \mathsf{p}_1 \quad \text{and} \quad \dot{\mathsf{x}}(0) = \mathbf{m}_0.$$

Recall that the derivative of a Bézier curve is

$$\dot{\mathsf{x}}(0) = n\sum_{i=0}^{n-1} \Delta\mathsf{b}_i B_i^{n-1}(t).$$

The two points are obtained right away, $\mathsf{x}(0) = \mathsf{b}_0 = \mathsf{p}_0$ and $\mathsf{x}(1) = \mathsf{b}_2 = \mathsf{p}_1$. Using the derivative formula we obtain the third point

$$\dot{\mathsf{x}}(0) = 2\Delta\mathsf{b}_0 = 2(\mathsf{b}_1 - \mathsf{b}_0) = \mathbf{m}_0, \quad \mathsf{b}_1 = \mathsf{p}_0 + \frac{1}{2}\mathbf{m}_0.$$

Thus the Hermite interpolating curve is

$$\mathsf{x}(t) = \mathsf{p}_0 B_0^2(t) + (\mathsf{p}_0 + \frac{1}{2}\mathbf{m}_0)B_1^2(t) + \mathsf{p}_1 B_2^2(t).$$